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Phase transitions in optimal betting strategies

L. DINIS[1], J. UNTERBERGER[2] and D. LACOSTE[3]

[1] *GISC - Grupo Interdisciplinar de Sistemas Complejos and Dpto. de Estructura de la Materia, Física Térmica y Electrónica, Universidad Complutense de Madrid - E-28040 Madrid, Spain*
[2] *Institut Elie Cartan, UMR CNRS 7502, Université de Lorraine - BP 239, F-54506 Vandoeuvre-lès-Nancy Cedex, France*
[3] *Gulliver Laboratory, UMR CNRS 7083, PSL Research University, ESPCI - 10 rue Vauquelin, F-75231 Paris Cedex 05, France*

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Abstract – Kelly's criterion is a betting strategy that maximizes the long-term growth rate, but which is known to be risky. Here, we find optimal betting strategies that gives the highest capital growth rate while keeping a certain low value of risky fluctuations. We then analyze the trade-off between the average and the fluctuations of the growth rate, in models of horse races, first for two horses then for an arbitrary number of horses, and for uncorrelated or correlated races. We find an analog of a phase transition with a coexistence between two optimal strategies, where one has risk and the other one does not. The above trade-off is also embodied in a general bound on the average growth rate, similar to thermodynamic uncertainty relations. We also prove mathematically the absence of other phase transitions between Kelly's point and the risk-free strategy.

Introduction. – Developed in 1956 by Bell Labs scientist John Kelly, Kelly's criterion applied the newly created field of information theory to gambling and investment [1]. Largely popularized in books [2], this criterion allows a gambler (or investment fund) to fix what proportion of bankroll should be risked on a given bet. It essentially exploits side information to maximize the expected geometric growth rate of a capital. This work was precursor to the growth optimal portfolio theory, which applied these ideas to capital market [3]. The ensemble of optimal investment strategies forms an efficient border [4], or equivalently a Pareto front [5,6], which is a term used in engineering and economics to call the set of designs that represent best trade-offs between different conflicting requirements.

Recently, there has been a surge of interest in applying insights from optimal gambling theory and economy to biology. Kelly's work led to an essential clarification of the concept of information value in biology [7,8], which was very helpful to understand strategies used by biological systems in a fluctuating environment. In particular the bet-hedging strategy turned out to be precisely an optimal strategy of the Kelly type [9,10].

Here, we focus on betting strategies of Kelly's type and draw inspiration from the field of Stochastic Thermodynamics, a recent branch of Thermodynamics with deep links to information theory, and with already several works specifically applied to gambling or betting problems [11–14]. A recent and an active line of research concerns the thermodynamic uncertainty relations [15–18], which capture important trade-offs in Thermodynamics. In this letter, we explore novel implications of these ideas for gambling models. We emphasize at this point that a background on Stochastic Thermodynamics is not required to understand this letter, since we only rely on basic notions of probability and optimization theory.

To gain insight into the trade-off present in gambling, we study the efficient border of Kelly's model, and we find that it extends to a region of negative growth, never discussed in the literature to our knowledge, corresponding to catastrophic betting strategies. Inspired by works on optimal protocols [19–21], and specifically on phase transitions among optimal protocols [22], we identify similar phase transitions in optimal betting strategies. We first prove such a result for uncorrelated races, and involving only two horses, which we then generalize to an arbitrary number of horses and to correlated races. In addition, we also give a general proof of the convexity in the most useful part of the front (positive part of the trade-off branch), which rules out the existence of further phase transitions on that branch.

Kelly's horse races. – Let us recall here the main features of Kelly's horse race [1,23]. This race involves M horses, which are numbered as $1, 2, \ldots, M$. The odds paid by the bookmaker when the horse x wins is o_x, and the probability for this to happen is p_x. A gambler can distribute his bets on the different horses, let b_x be the fraction of the bet set on horse x, so that $\sum_{x=1}^{M} b_x = 1$. For all x, $b_x > 0$, because the gambler bets on all horses but only makes money from the horse x that wins.

A key feature of the model is that this dynamics is repeated, since all the money gained in one race is reinvested in the next race. Thus, the capital C_{N+1} of the gambler after $N+1$ races is related to his capital after N races, C_N, by the expression

$$C_{N+1} = o_x b_x C_N, \quad \text{with probability } p_x. \tag{1}$$

The important quantity is the long-term growth rate of the capital which has the form

$$\lim_{N\to\infty} \frac{1}{N} \ln C_N = \sum_x p_x \ln(o_x b_x), \tag{2}$$

where the equality follows from the law of large numbers. Let us introduce the random variable $W_x = \ln(o_x b_x)$ which describes the contribution of horse x to this growth rate. Its average with respect to the probability density p_x, is the long-term growth rate denoted $\langle W \rangle$.

Kelly's strategy is defined from the optimization of this average growth rate over the betting strategy defined by b_x. A simple calculation given the constraint $\sum_x b_x = 1$ leads to the proportional betting strategy $b_x = p_x$. This particular solution is independent of the odds o_x, but if there was a track take, the optimal solution would depend on both o_x and p_x [1].

Games of this type can be easily simulated in a computer using a random number generator to choose a winning horse for each race according to probability distribution p_x and using eq. (1) to compute gambler's capital[1]. The growth of the capital is exponential and Kelly's strategy dominates on long times all non-optimal strategies as shown in fig. 1.

A central result of Stochastic Thermodynamics, namely fluctuation relations, can be obtained in a few steps for this model [14]. Using the definition of W, and given that $b = p$ for Kelly's strategy, we obtain

$$\langle e^{-W} \rangle = \sum_x p_x \frac{1}{o_x p_x} = \sum_x \frac{1}{o_x} = 1, \tag{3}$$

where in the last equality, we have used the normalization of the distribution $r_x = 1/o_x$ valid when there is no track take (fair odds). By Jensen's inequality, eq. (3) implies $\langle W \rangle \geq 0$, which also follows from $\langle W \rangle = D(p \mid r) \geq 0$ where $D(p \mid r)$ denotes the Kullback-Leibler (KL) divergence between the distributions p and r [23]. This fluctuation relation (3) can be generalized for an arbitrary strategy of the gambler, not necessarily that of Kelly, and when the odds are not necessarily fair, by introducing the decomposition $\tilde{W}_x = W_x + I_x$, where $W_x = \ln(o_x b_x)$ as above, $\tilde{W}_x = \ln(o_x p_x)$ and $I_x = \ln(p_x/b_x)$. In this way, $\tilde{W}$ represents the growth rate of the gambler according to Kelly's strategy and I measures the difference between the gambler's strategy and that of Kelly's in a KL sense, since $\langle I \rangle = D(p \mid b)$. We have then

$$\langle e^{-\tilde{W}} \rangle = \langle e^{-W-I} \rangle = \Lambda, \tag{4}$$

with $\Lambda = \sum_x 1/o_x$. In the same way that eq. (3) is the analog of Jarzynski equality, eq. (4) is similar to its generalization for absolutely irreversible processes [24]. By Jensen's inequality, the inequality $\langle W \rangle \geq -\langle I \rangle - \ln \Lambda$, follows which reduces to $\langle W \rangle \geq 0$ in the particular case of Kelly's strategy with fair odds. Note that in the general case, $\langle W \rangle$ can *a priori* be of any sign.

[1]See Supplementary Material `Supplementarymaterial.pdf` (SM) for details on simulations, on the exact solution for two horses, and on the analysis of the Pareto front.

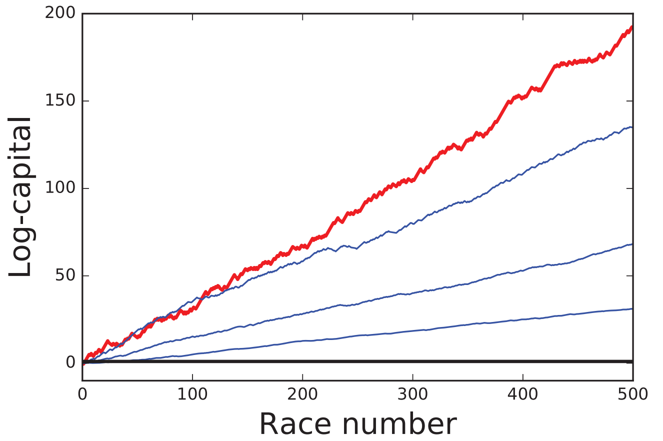

Fig. 1: Logarithm of the capital of the gambler *vs.* the number of races, for the optimal strategy (Kelly's) (thick red line) and for a selection of three non-optimal strategies (thin blue lines).

Mean-variance trade-off: choice of utility function. – Kelly's strategy focuses on the maximization of the growth rate at the price of overlooking risk. Although bankruptcy is absent in Kelly's scenario because the growth of the capital is geometric instead of arithmetic, the fluctuations of the capital are large as shown in fig. 1. This problem has been widely recognized in the gambling community. In practice gamblers and investors know that optimal Kelly can be "too risky"; and that "fractional Kelly" should be preferred, which deviates from the optimal solution but reduces the effective variance of the stochastic growth [3].

In the same spirit, we study here the optimal betting strategy that gives the highest capital growth rate while keeping a certain low value of risky fluctuations and analyze the corresponding trade-off between risk and gain. A similar idea is behind the mean-variance analysis introduced by Markowitz optimization [4]. In contrast with Markowitz optimization however, which considers the mean and variance of the capital return in one race, we consider here the mean and the variance of the (long-term) growth rate of the capital after many races. This important conceptual difference allows us to recover Kelly's point as a special case of our analysis, whereas Kelly's point could not appear as a limiting case of Markowitz's optimization for this reason. Hence, our utility function is a linear combination of the mean and standard deviation of the growth rate, namely $\langle W\rangle$ and σ_W:

$$\tilde{J} = \alpha\langle W\rangle - (1-\alpha)\sigma_W, \tag{5}$$

with $0 \leq \alpha \leq 1$. In practice, we use the modified utility function

$$J = \alpha\langle W\rangle - (1-\alpha)\sigma_W + \lambda\sum_x b_x, \tag{6}$$

where λ is a Lagrange multiplier associated to the normalization of the bets. An optimization of J with respect to b_x leads to $\lambda = -\alpha$. By reporting this into eq. (6), the optimal bets b_x are solutions of

$$p_x - b_x = \frac{\gamma}{\sigma_W} p_x[\ln(o_x b_x) - \langle W\rangle], \tag{7}$$

where $\gamma = (1-\alpha)/\alpha$. As expected, when $\alpha = 1$ ($\gamma = 0$), we recover the proportional betting of Kelly's strategy, which maximizes $\langle W\rangle$. Instead when $\alpha = 0$ ($\gamma \to \infty$), we obtain the *null strategy* also called the risk-free strategy, because in this case $\langle W\rangle = \sigma_W = 0$. Between these two values, the strategy of the gambler is described as *mixed* since it combines aspects associated to the optimization of $\langle W\rangle$ and σ_W.

Exact solution for two horses. – Before embarking on the full problem with an arbitrary number of horses, it is instructive to analyze the fully solvable case of two horses. Let the probability that the first horse wins (respectively, loses) be p (respectively, $1-p$); the bet and the odd on the first (respectively, second) horse are b and $1/r$ (respectively, $1-b$ and $1/(1-r)$) and let us introduce the parameter $\sigma = \sqrt{p(1-p)}$.

From the optimization of J, we obtain the optimal strategy $b^{\pm}$:

$$b^{\pm} = p \pm \gamma\sigma, \tag{8}$$

where the + (respectively, −) sign corresponds to an overbetting (respectively, underbetting) strategy with respect to Kelly's strategy where $b = p$.

By reporting the optimal bet given by eq. (8) into the expression of J, one obtains the efficient border. As shown in fig. 2, this border has two branches which meet at Kelly's point. When $p < r$ the lower blue solid line is the trade-off branch associated with b^+, while the upper red solid line is the non–trade-off branch, associated with b^-.

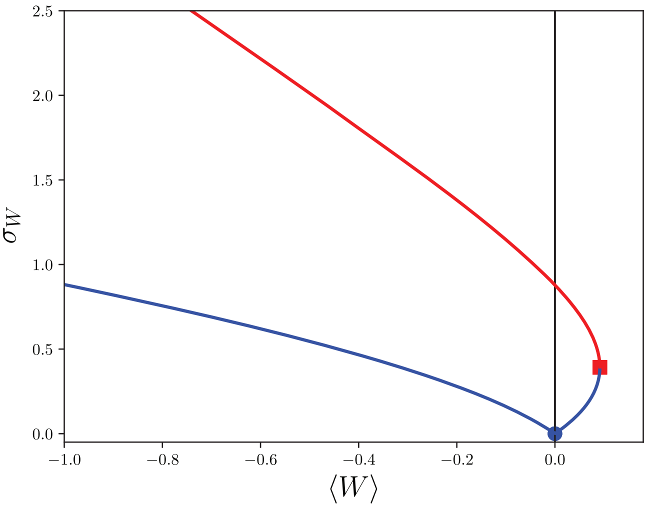

Fig. 2: Trade-off branch (lower blue solid line) and non–trade-off branch (upper red solid line) in the plane ($\langle W\rangle$, σ_W) for two horses and for the parameters ($p = 0.2, r = 0.4$). The two branches meet at the red square (Kelly's strategy), and the blue circle represents the null strategy.

The roles of b^- and b^+ exchange when instead $p > r$. Let us first focus on the region where $\langle W\rangle \geq 0$.

We find that the slope of the Pareto border is

$$\left.\frac{d\sigma_W}{d\langle W\rangle}\right|_{\gamma} = \frac{\sigma}{p-b}, \tag{9}$$

where b is equal to b^- when $r < p$ (see SM). Therefore the slope of the Pareto border is infinite at Kelly's point where $b^{\pm} = p$; while it reaches a finite value near the null strategy, namely

$$\left.\frac{d\sigma_W}{d\langle W\rangle}\right|_{\gamma_c} = \frac{1}{\gamma_c} = \frac{\sigma}{|p-r|}. \tag{10}$$

This signals a phase transition at this critical value γ_c, where the optimal strategy changes from the null strategy to a mixed strategy. As a result, the optimal J *vs.* γ changes from zero when $\gamma \geq \gamma_c$ (null strategy) to a non-zero value when $\gamma \leq \gamma_c$ (mixed strategy). For two horses, such a plot is similar to what is shown for three horses in the inset of fig. 3.

To prove the existence of the phase transition, we have checked that the border is convex near the null strategy. It is indeed the case since

$$\left.\frac{d^2\sigma_W}{d\langle W\rangle^2}\right|_{\gamma=\gamma_c} = \frac{r(1-r)}{\sigma^2\gamma_c^3} > 0. \tag{11}$$

In the rest of this paper, we now focus on the general case for an arbitrary number of horses.

Numerical results. – Let us now explain how to obtain the Pareto front from a numerical optimization of the utility function using a simulated annealing algorithm, as illustrated in fig. 3 for the case of three horses. Similarly

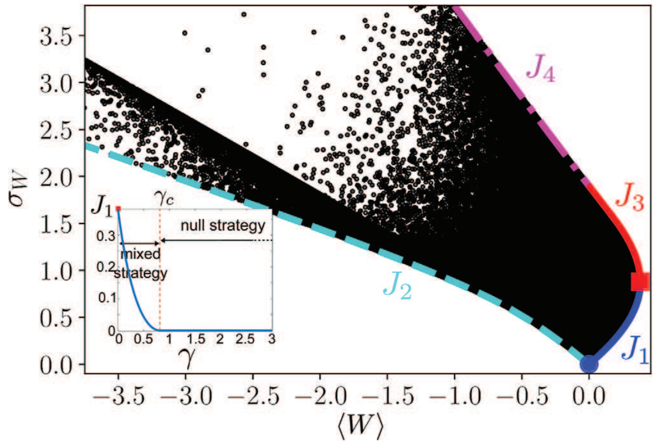

Fig. 3: Pareto borders for 3horses obtained from numerical optimization of the utility functions J_1, J_2, J_3 and J_4 (colored solid lines), together with a cloud of points generated by randomly choosing bets satisfying all relevant constraints. Parameters are $p_1 = 0.2$, $p_2 = 0.6$, $r_1 = 0.4$ and $r_2 = 0.2$ for the first two horses. Inset: J_1 *vs.* γ along the trade-off branch (*i.e.*, on the dark blue border).

to the case of two horses case, the lower and upper branch correspond to different optimization problems. The lower branch is formed by bets that maximize the growth rate $\langle W\rangle$ for a given value of the fluctuations σ_W, whereas the upper branch corresponds to maximal fluctuations σ_W for a given value of the growth rate $\langle W\rangle$.

For the lower branch, there are two regions where $\langle W\rangle$ is either positive or negative. In the former case, the front is convex and can be recovered by the maximization of the utility function $J = J_1$ defined in eq. (5). In contrast, in the negative $\langle W\rangle$ region, the front is concave and a different strategy is needed. Following [22], we use a quadratic objective function

$$J_2 = -(\langle W\rangle - W_0)^2 - k\sigma_W. \tag{12}$$

We use a global minus sign in order to keep the same maximization procedure, although we wish in fact to minimize both the value of σ_W and the distance to a target value W_0 for the growth rate. By varying the target value W_0 from 0 to a sufficiently negative value we can draw the negative lower branch. Parameter k weighs the importance between the constraint of $\langle W\rangle$ being close to W_0 or minimizing the value of the fluctuations. We took $k = 0.5$ although other moderate values would do.

Similarly, the upper branch with positive $\langle W\rangle$ is concave and corresponds to the maximization of the objective function

$$J_3 = \alpha\langle W\rangle + (1-\alpha)\sigma_W, \tag{13}$$

where the plus sign before σ_W now ensures the maximization of the fluctuations in contrast with the lower branch case. The upper branch with negative $\langle W\rangle$ appears almost straight for large negative values of $\langle W\rangle$. Thus, although J_3 could still be used there, further numerical precision can be achieved by using a modified objective function

$$J_4 = -(\langle W\rangle - W_0)^2 + k\sigma_W, \tag{14}$$

where again the plus sign in front of σ_W corresponds to the maximization of fluctuations.

General conclusions can also be obtained for this model near special points. Near Kelly's point, we find that the slope of the Pareto border is always vertical. This means that in practice if one is willing to sacrifice a small amount of the average growth rate, one can lower the fluctuations significantly, thereby accessing "safer" strategies such as the blue curves in fig. 1. Near the null strategy, we find a similar phase transition as in the two horses case, which we now analyze in more details.

Mean-variance trade-off: bounds. – We recall that $r_x := 1/o_x$ and we assume a fair game for which $\sum_x r_x = 1$. Then let $q_x := r_x/p_x$, so that the first two moments of q are $\langle q\rangle = 1$ and $\sigma_q^2 := \langle q^2\rangle - \langle q\rangle^2 = \langle q^2\rangle - 1$. Let us focus on the branch of positive $\langle W\rangle$. In this case, we find the following inequality:

$$\sigma_W \geq \frac{\langle W\rangle}{\sigma_q}, \tag{15}$$

which has a similar structure as thermodynamic uncertainty relations [15,25], and which captures a general trade-off between the mean and the variance of the growth rate.

The proof goes as follows: we consider the quantity $\sigma_q^2\sigma_W^2$, since $\sigma_q^2 = \langle q^2\rangle - 1$, we have using the Cauchy-Schwarz inequality

$$\begin{aligned}\sigma_q^2\sigma_W^2 &= \langle (q-1)^2\rangle\langle (W-\langle W\rangle)^2\rangle,\\ &\geq \langle (q-1)(W-\langle W\rangle)\rangle^2;\\ &\geq (\langle qW\rangle - \langle W\rangle)^2,\end{aligned} \tag{16}$$

Now since $\langle qW\rangle = \sum_x r_x/b_x \log(b_x/r_x) = -D(r|b) \leq 0$, then eq. (15) follows. This inequality is saturated when $b_x = r_x$, which corresponds to the null strategy.

Similar inequalities can be derived using instead other relevant Kullback-Leibler divergences, such as $D(b|p)$ or $D(r|p)$. To exploit the first divergence, we introduce the ratio $s_x = b_x/p_x$ which is also a normalized probability distribution similar to q, with a second moment σ_s^2. Then, following the same steps, we obtain an inequality for the quantity I introduced in eq. (3):

$$\sigma_I \geq \frac{\langle I\rangle}{\sigma_s}, \tag{17}$$

which is saturated when $b_x = p_x$, *i.e.* for Kelly's strategy. To exploit the second divergence, we now use the quantity $\tilde{W}$, and we obtain the inequality

$$\sigma_{\tilde{W}} \geq \frac{\langle \tilde{W}\rangle}{\sigma_q}, \tag{18}$$

which is saturated when $p_x = r_x$. Note that eqs. (17) and (18) provides new bounds that complement the inequalities $\langle I\rangle \geq 0$ and $\langle \tilde{W}\rangle \geq 0$ obtained previously.

Phase transition in optimal strategies. – In order to prove that there are no tighter bounds of this type, we carry out a perturbation calculation near the null strategy using the vector ϵ_x:

$$o_x b_x = \frac{b_x}{r_x} = 1 + \varepsilon_x. \tag{19}$$

To ensure that $\vec{b}$ is still a probability measure, we require that the column vector $\vec{\varepsilon} = (\varepsilon_x)_x$ lies on the hyperplane $(\vec{r}, \vec{\varepsilon}) = \sum_x r_x \varepsilon_x = 0$.

By evaluating $\langle W \rangle$ and σ_W to first order in $\vec{\varepsilon}$, we find that $\sigma_W \sim \langle W \rangle / \gamma_c$, with

$$\gamma_c = \sigma_q, \tag{20}$$

an expression which we can be checked by plotting a zoom of the Pareto border near the null strategy (see SM). The evaluation of the second order derivative at the null strategy on the Pareto border requires a calculation to second order in $\vec{\varepsilon}$, which gives

$$\left.\frac{\mathrm{d}^2\sigma_W}{\mathrm{d}\langle W \rangle^2}\right|_{\gamma=\gamma_c} = \frac{C}{\gamma_c^5}, \tag{21}$$

where $C = \langle q^3 \rangle - \langle q^2 \rangle^2$ (see SM). By Cauchy-Schwarz again, it follows that $\langle q^2 \rangle^2 = \langle q^{3/2} q^{1/2} \rangle^2 \leq \langle q^3 \rangle$, thus $C \geq 0$, with equality iff $p_x = r_x$.

In the particular case of two horses, it is straightforward to check that the expression of γ_c given in eq. (10) and that of the second derivative in eq. (11) are recovered from eqs. (20), (22). These calculations show that there is always a phase transition in this model near the null strategy for an arbitrary number of horses in the region of positive $\langle W \rangle$. A similar calculation shows that the slope has the opposite value on the other side in the region of negative $\langle W \rangle$.

Shape of the front: general results. –

Large negative growth rate. In the regions of the phase diagram corresponding to negative values of $\langle W \rangle$, the Pareto front is open. Namely, the growth rate diverges because it is evaluated on some $b_x \to 0$. Easy computations shows that points in the $(\langle W \rangle, \sigma_W)$ plane satisfy asymptotically $\langle W \rangle \to -\infty$ and $\sigma_W / \langle W \rangle \to -\sqrt{(1-P')/P'}$ when bets $b_{x'} \to 0$ for $x' \in X'$ with $P' := \sum_{x' \in X'} p_{x'}$. The smallest slope (lower front), is obtained by putting all the bets on the horse x^* which has the least chances to win; this is the worst strategy.

Lower front: positive growth rate. In order to decide whether other phase transitions are possible in this model, we now study the convexity of the front near any point. More precisely, we define the front as the extremum locus of the functional

$$\begin{aligned}\tilde{J}_{m^*}(b; \lambda, \mu) := \langle W^2 \rangle + \lambda(\langle W \rangle - m^*) \\ +2\mu(\sum_x b_x - 1),\end{aligned} \tag{22}$$

where λ, μ are Lagrange multipliers fixing $\langle W \rangle$ and implementing the bet normalization constraint. The procedure is equivalent to extremizing the variance for a given average value m^*. The null gradient condition $D\tilde{J}_{m^*}(b; \lambda, \mu) = 0$ defines (b, λ, μ) as an implicit function $f(m^*)$ of m^*. The gradient of f, which is the Hessian of $\tilde{J}_{m^*}$, may be inverted with some efforts, yielding by the implicit function theorem the slope $\mathrm{d}\sigma_W/\mathrm{d}\langle W \rangle = \mathrm{d}\sigma_W/\mathrm{d}m^*$ and then finally, the second derivative $\mathrm{d}^2\sigma_W/\mathrm{d}\langle W \rangle^2$ in terms of μ (proportional to the inverse of the Pareto slope parameter γ) and averaged functionals of b_x/p_x. Explicit formulas given in the SM have been checked numerically. One can then prove in whole generality that the part of the lower front between the null strategy and Kelly's strategy is convex, turning to concave in some neighborhood of the null strategy when $\langle W \rangle < 0$, and some neighborhood of Kelly's strategy on the upper front, as confirmed numerically in fig. 2 and fig. 3 in the case of two and three horses. Note that this calculation does not exclude the possibility of other phase transitions in other parts of the front.

Correlated races. – As a variation on Kelly's horse races, we now assume that the races are no longer independent but follow from an ergodic Markov process defined by the conditional probability $p_{x\,|\,y}$, which represents the probability that the horse x wins if the previous horse that won the race was horse y. Let the bets be also conditional and defined by $b_{x\,|\,y}$ such that $\sum_x b_{x\,|\,y} = 1$. The odds denoted by $o_x = 1/r_x$ are assumed to be fair $\sum_x r_x = 1$. The average growth rate $\langle W \rangle$ now takes the following form

$$\langle W \rangle = \lim_{N\to\infty} \langle W_N \rangle = \sum_{x,y} p_{x\,|\,y} \bar{p}_y \ln(b_{x\,|\,y} o_x), \tag{23}$$

where $\bar{p}_y$ denote the unique steady state probability of the races. By optimizing $\langle W \rangle$ with respect to $b_{x|y}$, we find that the optimal strategy is still proportional betting with now $p_{x|y} = b_{x|y}$. This is the new Kelly's strategy for this case.

On the trade-off branch, the relevant utility function is

$$J = \alpha \langle W \rangle - (1-\alpha)\sigma_W + \sum_y \lambda_y \sum_x b_{x|y}, \tag{24}$$

where λ_y are Lagrange multipliers associated to the normalization of the bets. The Pareto borders are shown in fig. 4. We observe numerically that when correlations are present the upper front for negative W becomes convex in some intermediate region. In that region, the border can not longer be described by J_3 and the use of J_4 is unavoidable.

The null strategy corresponds to the condition that for any x, y, $b_{x|y} = r_x$, in which case both the average growth rate and its variance are zero. An expansion with respect to that strategy can be carried as before. The q distribution is now defined as $q_{x\,|\,y} = r_x / p_{x\,|\,y}$, which is a

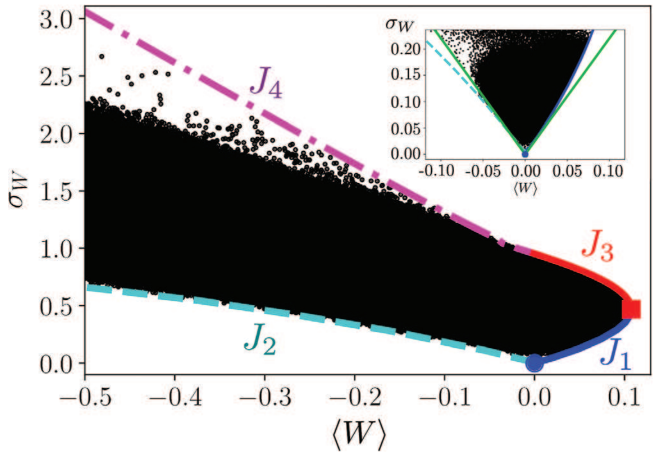

Fig. 4: Same plot as in fig. 3 but for the case of for 3 horses in the presence of correlations between the races. Parameters are detailed in the SM. Inset: zoom near the null strategy together with predictions from linear approximation.

probability distribution because

$$\langle q \rangle = \sum_{xy} p_{x\,|\,y} \bar{p}_y \frac{r_x}{p_{x\,|\,y}} = \sum_{xy} \bar{p}_y r_x = 1. \tag{25}$$

Its second moment is now $\langle q^2 \rangle = \sum_{xy} p_{x|y} \bar{p}_y q^2_{x|y}$. Except for this modification, the critical γ takes the same form as in eq. (20), which is numerically tested in the inset of fig. 4.

An inequality similar to eq. (15) can also be obtained in the case of correlated races because in this case the conditional bets $b_{x\,|\,y}$ are still a probability distribution $\sum_x b_{x\,|\,y} = 1$, and therefore following the same steps, the positivity of $D(r|b)$ leads to a similar result. In fact, the normalization of q is equivalent to a fluctuation relation generalizing eq. (4) [14]. because in that case

$$\langle e^{-W} \rangle = \sum_{xy} p_{x\,|\,y} \bar{p}_y \frac{1}{p_{x\,|\,y} o_x} = 1, \tag{26}$$

while

$$\langle e^{-W-I} \rangle = \Lambda, \tag{27}$$

holds in the general case for an arbitrary strategy with $I_{x|y} = \ln(p_{x|y}/b_{x|y})$.

Conclusion. – In this work, we have derived general fluctuation relations for betting models of Kelly's type, and a bound on the average capital growth rate, similar to thermodynamic uncertainty relations. This bound captures the classic trade-off between average growth rate and risk, which plays a central role in money investment [26]. In models with repetitive investment dynamics, all utility functions become under suitable conditions equivalent to a utility function with a log mean-variance form [3], which is the form considered here. This suggests that our work should be applicable to a broad class of econophysics models, for which log utility functions are used.

In our work, we have identified a phase transition between the null strategy and a mixed strategy, and we have shown that there is no other phase transition between the null strategy and Kelly's point due to the convexity of the lower front. We have also illustrated how to handle non-convex utility functions, an important issue for applications to machine learning [27].

The explicit analytical expressions which we have obtained for the slope and curvature of the front at any point could be used to move directly along the front, as an alternative to the involved optimization algorithm used here. It would be also interesting to explore more systematically how additional constraints affect the efficient border. The question of adaptative optimization of the bets, where possible non-Markovian or non-ergodic features could arise, is a rich inference problem worth pursuing [10]. Finally, we hope that this framework could open news research directions on evolutionary trade-offs and Pareto optimality in biology [5,6].

* * *

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