Optics of a Faraday-active Mie sphere

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We present an exact calculation for the scattering of light from a single sphere made of a Faraday-active material into first order of the external magnetic field. When the size of the sphere is small compared with the wavelength, the known T matrix for a magneto-active Rayleigh scatterer is found. We address the issue of whether there is a so-called photonic Hall effect—a magneto-transverse anisotropy in light scattering—for one Mie scatterer. In the limit of geometrical optics, we compare our results with the Faraday effect in a Fabry–Perot etalon. © 1998 Optical Society of America [S0740-3232(98)01906-1]

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1. INTRODUCTION

There are several reasons why one wishes to understand light scattering from a dielectric sphere made of magneto-active material. Single scattering is the building block for multiple scattering. Many experiments have been done with diffuse light in a magnetic field.[1–3] Though qualitatively very useful, a theory for pointlike scatterers in a magnetic field, as first developed by MacKintosh and John[4] and later refined by van Tiggelen *et al.*,[5] does not always describe observations quantitatively, for the obvious reason that experiments do not contain small scatterers. In this paper light scattering from one sphere of any size in a homogeneous magnetic field is addressed.

The model of Rayleigh scatterers has been used successfully to describe specific properties of multiple light scattering in magnetic fields, such as coherent backscattering and the photonic Hall effect (PHE). In Section 2 we present the perturbation approach on which our work is based, which allows us in Section 3 to compute the T matrix for this problem. With this tool we are able to answer in Section 4 the issue concerning the PHE.

2. PERTURBATION THEORY

We consider the light scattered by one dielectric sphere made of a Faraday-active medium embedded in an isotropic medium with no magneto-optical properties, using a perturbative approach to first order in the magnetic field.

In this paper we set $c_0 = 1$. In a magnetic field the refractive index of the sphere is a tensor of rank 2. This index depends on the distance to the center of the sphere r, which is given a radius a by means of the Heaviside function $\Theta(r - a)$, which equals 1 inside the sphere and 0 outside:

$$\varepsilon(\mathbf{B}, \mathbf{r}) - \mathbf{I} = [(\varepsilon_0 - 1)\mathbf{I} + \varepsilon_F \mathbf{\Phi}]\Theta(|\mathbf{r}| - a). \quad (1)$$

In this expression $\varepsilon_0 = m^2$ is the value of the normal isotropic dielectric constant of the sphere of relative index of refraction m (which can be complex), and $\varepsilon_F = 2mV_0/\omega$ is a coupling parameter associated with the amplitude of the Faraday rotation (V_0 being the Verdet constant and ω the frequency). We introduce the antisymmetric Hermitian tensor $\Phi_{ij} = i\epsilon_{ijk}B_k$. Except for ε_F, the Mie solution depends on the dimensionless size parameters $x = ka$ with $k = \omega/c_0$ and $y = mx$. In this paper we restrict our investigation to nonabsorbing media, so that m and ε_F are real valued.

Noting that the Helmholtz equation is formally analogous to a Schrödinger equation with potential $\mathbf{V}(\mathbf{r}, \omega) = [\mathbf{I} - \varepsilon(\mathbf{B}, \mathbf{r})]\omega^2$ and energy ω^2, we obtain the T matrix with the following Born series[6]:

$$\mathbf{T}(\mathbf{B}, \mathbf{r}, \omega) = \mathbf{V}(\mathbf{r}, \omega) + \mathbf{V}(\mathbf{r}, \omega) \cdot \mathbf{G}_0 \cdot \mathbf{V}(\mathbf{r}, \omega) + \ \dots \ . \quad (2)$$

Here $\mathbf{G}_0(\omega, \mathbf{p}) = 1/(\omega^2\mathbf{I} - p^2\mathbf{\Delta}_p)$ is the free Helmholtz Green's function, and $(\Delta_p)_{ij} = \delta_{ij} - p_i p_j/p^2$. We call $\mathbf{T}^0$ the part of $\mathbf{T}$ that is independent of the magnetic field and $\mathbf{T}^1$ the part of the T matrix that is linear in $\mathbf{B}$. It follows from Eq. (2) that

$$\mathbf{T}^1 = (1 - \varepsilon_0)\varepsilon_F \Theta \mathbf{T}^0 \cdot \mathbf{\Phi} \cdot (\mathbf{I} + \mathbf{G}_0 \cdot \mathbf{T}^0). \quad (3)$$

We need to introduce the unperturbed eigenfunctions $\Psi^{\pm}_{\sigma,\mathbf{k}}(\mathbf{r})$ of the conventional Mie problem. These eigenfunctions represent the electric field at the point $\mathbf{r}$ for an incident plane wave $|\sigma, \mathbf{k}\rangle$ along the direction $\mathbf{k}$ with an helicity σ. This eigenfunction is "outgoing" for $\Psi^{+}_{\sigma,\mathbf{k}}$ and "ingoing" for $\Psi^{-}_{\sigma,\mathbf{k}}$, according to the definition of the outgoing and the ingoing free Helmholtz Green's function:

$$|\Psi^{\pm}_{\sigma,\mathbf{k}}\rangle = (\mathbf{I} + \mathbf{G}_0^{\pm} \cdot \mathbf{T}^0)|\sigma,\mathbf{k}\rangle.$$

For our free Helmholtz Green's function this implies[6]

$$\Psi^{-*}_{\sigma,-\mathbf{k}}(\mathbf{r}) = (-1)^{1+\sigma}\Psi^{+}_{-\sigma,\mathbf{k}}(\mathbf{r}). \quad (4)$$

We denote by $\mathbf{k}$ the incident direction and by $\mathbf{k}'$ the scattered direction. With this notation it is possible to obtain from Eq. (3)

$$\mathbf{T}^1_{\mathbf{k}\sigma,\mathbf{k}'\sigma'} = \varepsilon_F \omega^2 \langle \Psi^{-}_{\sigma,\mathbf{k}} | \Theta \mathbf{\Phi} | \Psi^{+}_{\sigma',\mathbf{k}'} \rangle. \quad (5)$$

This equation can also be obtained from standard first-order Rayleigh–Schrödinger perturbation theory.[7]

Two important symmetry relations are satisfied by our T matrix. The first one is the parity, and the second one is the reciprocity:

$$\mathbf{T}_{-\mathbf{k}-\sigma,-\mathbf{k}'-\sigma'}(\mathbf{B}) = \mathbf{T}_{\mathbf{k}\sigma,\mathbf{k}'\sigma'}(\mathbf{B}), \quad (6)$$

$$\mathbf{T}_{-\mathbf{k}'\sigma',-\mathbf{k}\sigma}(-\mathbf{B}) = \mathbf{T}_{\mathbf{k}\sigma,\mathbf{k}'\sigma'}(\mathbf{B}). \quad (7)$$

3. *T* MATRIX FOR MIE SCATTERING

To separate the radial and the angular contributions in Eq. (5), we expand the Mie eigenfunction $\Psi^{+}_{\sigma,\mathbf{k}}$ on the basis of the vector spherical harmonics[6]:

$$\Psi^{+}_{\sigma,\mathbf{k}}(\mathbf{r}) = \frac{2\pi}{\rho}\, i^{J+1}\mathbf{Y}^{\lambda'}_{JM}(\hat{\mathbf{r}}) f^{J}_{\lambda'\lambda}(r)\mathbf{Y}^{\lambda*}_{JM}(\hat{\mathbf{k}}) \cdot \chi'_{\sigma}. \quad (8)$$

In this definition $\rho = kmr$, the χ'_{σ} are the eigenvectors of the spin operator in the circular basis associated with the direction $\mathbf{k}$, and implicit summation over the repeated indices J, M, λ, and λ' has been assumed (the indices λ and λ' for the components of the field can take three values, one being longitudinal and two being perpendicular to the direction of propagation).

Because of the presence of the function Θ in Eq. (5), we have to consider only the field inside the sphere, whose main features are contained in the radial function $f^{J}_{\lambda'\lambda}(r)$. This matrix $f^{J}_{\lambda'\lambda}$ is known in terms of the transmission coefficients c_J and d_J of Ref. 8, and of the Ricatti–Bessel function $u_J(\rho)$. We found the following expression:

$$\mathbf{f}^J(\rho) = m\begin{bmatrix} -iu'_J(\rho)c_J & 0 & 0 \\ -iu'_J(\rho)c_J[J(J+1)]^{1/2}/\rho & 0 & 0 \\ 0 & 0 & u_J(\rho)d_J \end{bmatrix}.$$

The three vectors χ_σ, which are similar to the χ'_σ but associated with the z axis, are a convenient basis for this problem since they are the eigenvectors of the operator $\mathbf{\Phi}$ with eigenvalue $-\sigma$, provided that we choose the z axis along $\mathbf{B}$, which we do below. Equation (5) is simplified by this choice, and the angular integration leads eventually to

$$\int \mathbf{Y}^{\lambda_1 *}_{J_1 M_1}(\hat{\mathbf{r}}) \cdot \mathbf{\Phi} \cdot \mathbf{Y}^{\lambda_2}_{J_2 M_2}(\hat{\mathbf{r}}) \mathrm{d}\Omega_r = \delta_{J_1 J_2}\delta_{M_1 M_2} Q_{\lambda_1\lambda_2}(J_1, M_1),$$

where $\mathbf{Q}$ is the matrix

$$\mathbf{Q}(J, M) = -MB\begin{bmatrix} \frac{1}{J(J+1)} & \frac{1}{\sqrt{J(J+1)}} & 0 \\ \frac{1}{\sqrt{J(J+1)}} & 0 & 0 \\ 0 & 0 & \frac{1}{J(J+1)} \end{bmatrix} \quad (9)$$

and B is the absolute value of the applied magnetic field. The linear dependence on the magnetic quantum number M can be expected for an effect like the Faraday rotation, affecting left and right circularly polarized light in an opposite way, similar to Zeeman splitting. The radial integration can be performed with a method developed by Bott *et al.*[9]:

$$\mathbf{T}^1_{\mathbf{k},\mathbf{k}'} = \frac{16\pi}{\omega}\,\mathrm{W}\sum_{J,M}(-M)[\mathscr{C}_J \mathbf{Y}^{e}_{J,M}(\hat{\mathbf{k}})\mathbf{Y}^{e*}_{J,M}(\hat{\mathbf{k}}') + \mathscr{D}_J \mathbf{Y}^{m}_{J,M}(\hat{\mathbf{k}})\mathbf{Y}^{m*}_{J,M}(\hat{\mathbf{k}}')], \quad (10)$$

with the dimensionless parameter

$$\mathrm{W} = V_0 B \lambda$$

and the coefficients

$$\mathscr{C}_J = -\frac{2c_J^{2*}|u_J|^2 y^3}{J(J+1)(y^2 - y^{*2})}\left(\frac{A_J^*}{y^*} - \frac{A_J}{y}\right), \quad (11)$$

$$\mathscr{D}_J = -\frac{2d_J^{2*}|u_J|^2 y^3}{J(J+1)(y^2 - y^{*2})}\left(\frac{A_J^*}{y} - \frac{A_J}{y^*}\right), \quad (12)$$

with $A_J(y) = u'_J(y)/u_J(y)$ and B the amplitude of the magnetic field directed along the unit vector $\hat{\mathbf{B}}$. Absorption in the sphere is still allowed. We consider the limiting case of a perfect dielectric sphere with no absorption $[\mathrm{Im}(m) \to 0]$. Using l'Hospital's rule in Eqs. (11) and (12), we immediately obtain for this case

$$\mathscr{C}_J = \frac{-c_J^{2*}u_J^2 y}{J(J+1)}\left[\frac{A_J}{y} - \frac{J(J+1)}{y^2} + 1 + A_J^2\right], \quad (13)$$

$$\mathscr{D}_J = \frac{-d_J^{2*}u_J^2 y}{J(J+1)}\left[-\frac{A_J}{y} - \frac{J(J+1)}{y^2} + 1 + A_J^2\right]. \quad (14)$$

A. *T* Matrix without Magnetic Field

For future use we need the on-shell T matrix of the conventional Mie problem.[8] It is given by a formula analogous to Eq. (10), where $\mathscr{C}_J$ and $\mathscr{D}_J$ are replaced by the Mie coefficients a_J and b_J and by $M = 1$. Because of rotational invariance of the scatterer, it is clear that the final result depends only on the scattering angle θ, which is the angle between $\mathbf{k}$ and $\mathbf{k}'$ (see Fig. 1). Therefore we obtain in the circular basis (associated with the indices σ and σ')

$$T^0_{\sigma\sigma'} = \frac{2\pi}{i\omega}\sum_{J\geqslant 1}\frac{2J+1}{J(J+1)}(a_J^* + \sigma\sigma' b_J^*) \times [\pi_{J,1}(\cos\theta) + \sigma\sigma'\tau_{J,1}(\cos\theta)]. \quad (15)$$

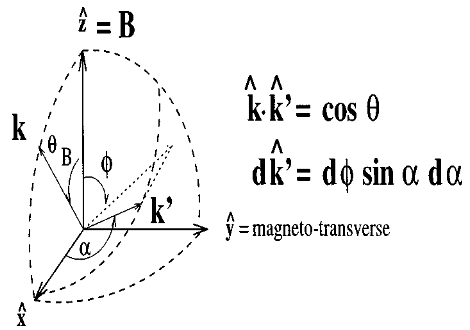

Fig. 1. Schematic view of the magneto-scattering geometry. Generally, θ denotes the angle between incident and outgoing wave vectors; ϕ is the azymuthal angle in the plane of the magnetic field and the y axis. The latter is by construction the magneto-transverse direction defined as the direction perpendicular to both the magnetic field and the incident wave vector. Angle α coincides with angle θ in the special but relevant case that the incident vector is normal to the magnetic field.

In this formula the polynomials $\pi_{J,M}$ and $\tau_{J,M}$ are defined in terms of the Legendre polynomials P_J^M by[8]

$$\pi_{J,M}(\cos\,\theta) = \frac{M}{\sin\,\theta} P_J^M(\cos\,\theta),$$

$$\tau_{J,M}(\cos\,\theta) = \frac{\mathrm{d}}{\mathrm{d}\theta} P_J^M(\cos\,\theta). \tag{16}$$

B. $\mathbf{T}^1_{\mathbf{kk}'}$ When $\hat{\mathbf{k}} \neq \hat{\mathbf{k}}'$

We have yet to express the vector spherical harmonics in Eq. (10) in terms of the natural angles of the problem in the presence of a magnetic field. The magnetic field breaks rotational invariance. Because $\mathbf{T}^1$ is linear in $\hat{\mathbf{B}}$, we can construct it by considering three special cases for the direction of $\hat{\mathbf{B}}$. If $\hat{\mathbf{k}} \neq \hat{\mathbf{k}}'$, we can decompose the unit vector $\hat{\mathbf{B}}$ in the nonorthogonal but complete basis of $\hat{\mathbf{k}}$, $\hat{\mathbf{k}}'$ and $\hat{\mathbf{g}} = \hat{\mathbf{k}} \times \hat{\mathbf{k}}'/|\hat{\mathbf{k}} \times \hat{\mathbf{k}}'|$, and this results in

$$\mathbf{T}^1_{\mathbf{kk}'} = \frac{(\hat{\mathbf{B}}\cdot\hat{\mathbf{k}})(\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}') - \hat{\mathbf{B}}\cdot\hat{\mathbf{k}}'}{(\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}')^2 - 1}\mathbf{T}^1_{\hat{\mathbf{B}}=\hat{\mathbf{k}}'} + \frac{(\hat{\mathbf{B}}\cdot\hat{\mathbf{k}}')(\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}') - \hat{\mathbf{B}}\cdot\hat{\mathbf{k}}}{(\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}')^2 - 1}\mathbf{T}^1_{\hat{\mathbf{B}}=\hat{\mathbf{k}}} + (\hat{\mathbf{B}}\cdot\hat{\mathbf{g}})\mathbf{T}^1_{\hat{\mathbf{B}}=\hat{\mathbf{g}}}, \tag{17}$$

with

$$T^1_{\sigma\sigma'}(\hat{\mathbf{B}} = \hat{\mathbf{k}}) = \frac{2\mathrm{W}}{\omega}\sum_{J\geqslant 1}\frac{2J+1}{J(J+1)}(-\sigma)(\mathscr{C}_J + \sigma\sigma'\mathscr{D}_J) \times [\pi_{J,1}(\cos\,\theta) + \sigma\sigma'\tau_{J,1}(\cos\,\theta)], \tag{18}$$

$$T^1_{\sigma\sigma'}(\hat{\mathbf{B}} = \hat{\mathbf{k}}') = \frac{2\mathrm{W}}{\omega}\sum_{J\geqslant 1}\frac{2J+1}{J(J+1)}(-\sigma')(\mathscr{C}_J + \sigma\sigma'\mathscr{D}_J) \times [\pi_{J,1}(\cos\,\theta) + \sigma\sigma'\tau_{J,1}(\cos\,\theta)], \tag{19}$$

$$T^1_{\sigma\sigma'}(\hat{\mathbf{B}} = \hat{\mathbf{g}}) = \frac{4i\mathrm{W}}{\omega}\sum_{\substack{J\geqslant 1\\ J\geqslant M>0}}\frac{2J+1}{J(J+1)}M\,\sin(M\theta) \times \frac{(J-M)!}{(J+M)!}(\sigma\sigma'\mathscr{C}_J + \mathscr{D}_J) \times [\pi^2_{J,M}(0) + \sigma\sigma'\tau^2_{J,M}(0)]. \tag{20}$$

C. Particular Case for T^1 for Forward Scattering

The treatment in Subsection 3.B becomes degenerate when $\hat{\mathbf{k}}$ and $\hat{\mathbf{k}}'$ are parallel, i.e., in forward scattering. In this case $\hat{\mathbf{B}}$ can still be expressed in a basis made of $\hat{\mathbf{k}}$ and of two vectors perpendicular to $\hat{\mathbf{k}}$. The contribution of these last two vectors must have the same form as in Eq. (20) for $\theta = 0$. Hence there is no such contribution, and we find

$$T^1_{\mathbf{k}\sigma,\mathbf{k}\sigma'} = \delta_{\sigma\sigma'}(\hat{\mathbf{B}}\cdot\hat{\mathbf{k}})(-\sigma)\frac{2\mathrm{W}}{\omega}\sum_{J\geqslant 1}(2J+1)(\mathscr{C}_J + \mathscr{D}_J). \tag{21}$$

In Fig. 2 we have plotted the real and the imaginary parts of this expression in units of W as a function of the size parameter x for $\sigma = -1$ and $\hat{\mathbf{B}} = \hat{\mathbf{k}}$. The forward-scattering amplitude has an important application in inhomogeneous media, namely, as the complex average dielectric constant.

D. Optical Theorem

We check our formula on energy conservation as expressed by the Optical Theorem[6]:

$$-\frac{\mathrm{Im}(T_{\mathbf{k}\sigma,\mathbf{k}\sigma})}{\omega} = \sum_{\sigma'}\int \mathrm{d}\Omega_{\mathbf{k}'}\frac{|T_{\mathbf{k}\sigma,\mathbf{k}'\sigma'}|^2}{(4\pi)^2}. \tag{22}$$

To first order in the magnetic field, the right-hand side of this equation equals

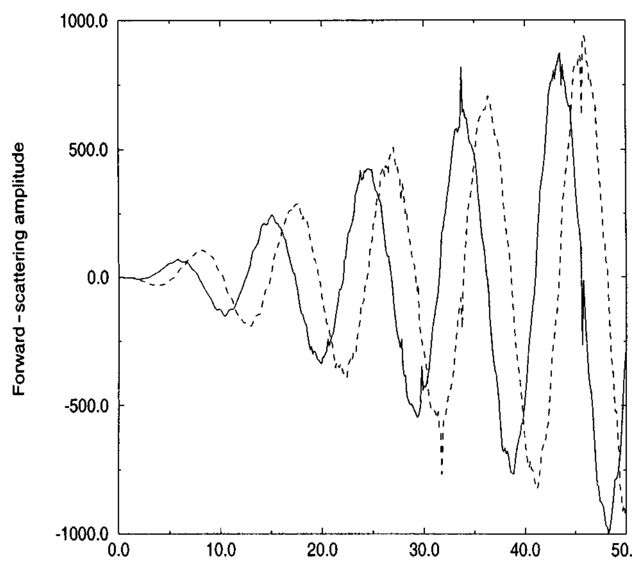

Fig. 2. Real part (solid curve) and imaginary part (dashed curve) of the magneto-forward scattering matrix $\mathbf{T}^1_{\mathbf{kk}}$ in the circular basis of polarization, plotted versus size parameter x for index of refraction $m = 1.33$ in units of $\mathrm{W} = V_0 B\lambda$.

$$\frac{1}{8\pi^2}\sum_{\sigma'}\int \mathrm{d}\Omega_{\mathbf{k}'}\ \mathrm{Re}(T^0_{\mathbf{k}\sigma,\mathbf{k}'\sigma'}T^{1*}_{\mathbf{k}\sigma,\mathbf{k}'\sigma'}).$$

If we assume that $\hat{\mathbf{B}}\|\hat{\mathbf{k}}$, we can compute this using Eqs. (18) and (15) and the following orthogonality relations for the polynomials $\pi_{J,1}$ and $\tau_{J,1}$ (which we denote π_J and τ_J)[6]:

$$\int \mathrm{d}(\cos\,\theta)[\pi_J(\cos\,\theta)\tau_K(\cos\,\theta) + \tau_J(\cos\,\theta)\pi_K(\cos\,\theta)] = 0,$$

$$\int \mathrm{d}(\cos\,\theta)[\pi_J(\cos\,\theta)\pi_K(\cos\,\theta) + \tau_J(\cos\,\theta)\tau_K(\cos\,\theta)] = \frac{2J^2(J+1)^2}{2J+1}\,\delta_{JK}. \quad (23)$$

The left-hand side of Eq. (22) is obtained from Eq. (21). The Optical Theorem gives us a relation between Mie coefficients, which we can actually prove from their definitions:

$$\mathrm{Re}[a_J^* c_J^2(2/i)] = \mathrm{Im}(c_J^2), \quad (24)$$

$$\mathrm{Re}[b_J^* d_J^2(2/i)] = \mathrm{Im}(d_J^2). \quad (25)$$

4. MAGNETO-TRANSVERSE SCATTERING

From our knowledge of the matrix $\mathbf{T}^1$ we can compute how the magnetic field affects the differential scattering cross section summed over polarizations as a function of the scattering angle. Only the diagonal part of this matrix in a basis of linear polarization will affect the scattering cross section, since we consider only terms to first order in the magnetic field. Therefore only the third contribution of Eq. (17) plays a role, which means that the effect will be maximum when $\hat{\mathbf{B}}\|\hat{\mathbf{k}}\times\hat{\mathbf{k}}'$ (the typical Hall geometry). Indeed, symmetry implies that the magneto-scattering cross section should be the product of $\det(\hat{\mathbf{B}}, \hat{\mathbf{k}}, \hat{\mathbf{k}}')$ and of some function that depends entirely on the scattering angle θ.

We choose to normalize this magneto-scattering cross section by the total scattering cross section in the absence of the magnetic field:

$$\frac{2\sum_{\sigma\sigma'}\ \mathrm{Re}(T^0_{\sigma\sigma'},\ T^{1*}_{\sigma\sigma'})}{\int_0^{2\pi}\mathrm{d}\phi\int_0^{\pi}\mathrm{d}\cos\,\theta\sum_{\sigma\sigma'}|T^0_{\sigma\sigma'}|^2} = -\sin\,\phi F(\theta). \quad (26)$$

The so-called PHE is a manifestation of a magnetically induced transverse current in the light transport that has similarities to the Hall effect, which is known for the transport of electrons. In an experiment on the PHE, one measures the difference in scattered light from two opposite directions: perpendicular both to the incident direction of light and to the applied magnetic field.[2] The PHE is a manifestation of the anisotropy of light scattering that is due to a magnetic field in the regime of multiple light scattering.

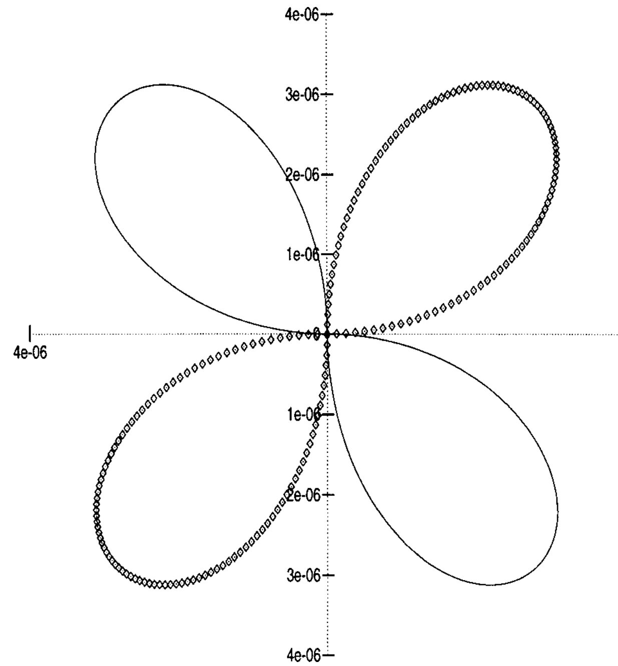

Fig. 3. Magneto-transverse scattering cross section $F(\theta)$ for a Rayleigh scatterer with index of refraction $m = 1.1$ and size parameter $x = 0.1$. Solid curve, positive correction; points, negative correction. The curve has been normalized by the parameter W. No net magneto-transverse scattering is expected in this case because the projections onto the y axis of these corrections cancel one another. Axis numbers in the format 4e-06 are equivalent to 4×10^{-6}.

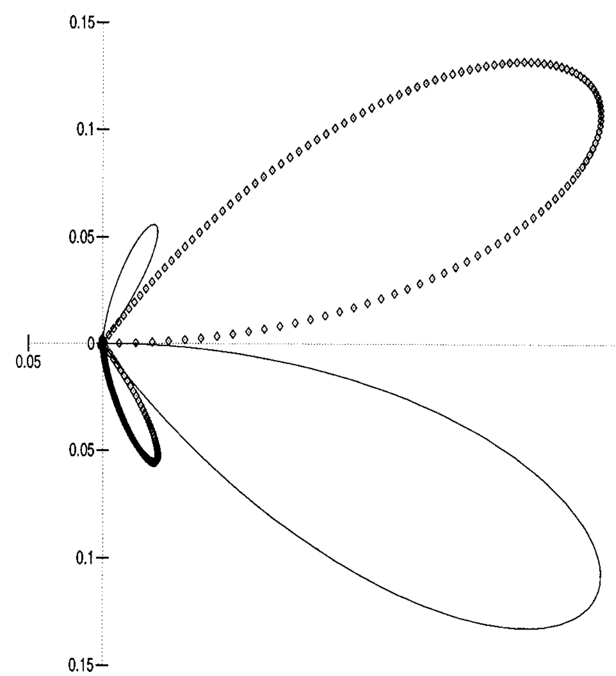

Fig. 4. Magneto-transverse scattering cross section $F(\theta)$ for a Mie scatterer of size parameter $x = 5$ and of index of refraction $m = 1.1$. The curve has been normalized by the parameter W. Solid curve, positive correction; points, negative correction. In this case a net magneto-transverse scattering is expected because the projections onto the y axis of these corrections do not cancel one another.

Although experiments have dealt with multiple scattering so far, it is interesting to see whether a net magneto-transverse scattering persists for only one single scatterer. In Figs. 3 and 4 the magnetic field is perpendicular to the plane of the figure, and the incident light is along the x axis. A typical measurement of the magneto-transverse scattered light is therefore associated with the projection of the curve onto the y axis, which we define as the magneto-transverse direction.

5. MAGNETO-TRANSVERSE SCATTERING AS A FUNCTION OF THE SIZE PARAMETER

Quantitatively, the transverse-light-current difference is associated with a summation of the magneto-scattering cross section over outgoing wave vectors and normalized to the total transverse light current. A schematic view of the geometry is displayed in Fig. 1. In our notation this is

$$\eta \equiv \frac{I_{\mathrm{up}}(B) - I_{\mathrm{down}}(B)}{I_{\mathrm{up}}(B=0) + I_{\mathrm{down}}(B=0)} = \frac{2\int_0^\pi \mathrm{d}\phi \int_0^\pi \mathrm{d}\cos\theta \sin\theta \sin\phi \sum_{\sigma\sigma'} \mathrm{Re}(T^0_{\sigma\sigma'} T^{1*}_{\sigma\sigma'})}{\int_0^\pi \mathrm{d}\phi \int_0^\pi \mathrm{d}\cos\theta \sin\theta \sin\phi \sum_{\sigma\sigma'} |T^0_{\sigma\sigma'}|^2}. \tag{27}$$

The factor $\sin\theta \sin\phi$ represents a projection onto the magneto-transverse direction $\hat{\mathbf{B}} \times \hat{\mathbf{k}}$, which is necessary since we are interested in the magneto-transverse light flux. In Fig. 5 we plotted this contribution as a function of the size parameter x for an index of refraction $m = 1.0946$ (the value in Ref. 2). Note the change of sign beyond $x = 1.7$, for which we do not yet have any simple explanation. In the range of the small size parameter, η exhibits an x^5 dependence.

A. Rayleigh Scatterers

For Rayleigh scatterers formulas (17)–(21) simplify dramatically because one needs to consider only the first partial wave of $J = 1$ and the first terms in a development in powers of y (since $y \ll 1$). From Eqs. (13) and (14) we find $\mathscr{C}_1 = -2y^3/m^2(2 + m^2)^2$ and $\mathscr{D}_1 = -y^5/45m^4$, so we can keep only $\mathscr{C}_1$ and drop $\mathscr{D}_1$ as a first approximation. Adding all contributions of Eqs. (17) and (15) and changing from a circular basis to a linear basis of polarization, we find

$$\mathbf{T}_{\mathbf{k},\mathbf{k}'} = \begin{bmatrix} t_0\hat{\mathbf{k}} \cdot \hat{\mathbf{k}}' + it_1\hat{\mathbf{B}} \cdot (\hat{\mathbf{k}} \times \hat{\mathbf{k}}') & it_1\hat{\mathbf{B}} \cdot \hat{\mathbf{k}} \\ -it_1\hat{\mathbf{B}} \cdot \hat{\mathbf{k}}' & t_0 \end{bmatrix}, \tag{28}$$

where $t_0 = -6i\pi a_1^*/\omega$ is the conventional Rayleigh T matrix and $t_1 = 6\mathscr{C}_1 \mathrm{W}/\omega$. This form agrees with the Rayleigh pointlike-scatterer model discussed in Ref. 5. We note that Eqs. (18) and (19) give off-diagonal contributions in Eq. (28), whereas Eq. (20) gives a diagonal contribution. This is a general feature that also persists beyond the regime of Rayleigh scatterers.

For a Rayleigh scatterer the magneto cross section of Fig. 3 exhibits symmetry, since the positive and negative lobes of the curve are of the same size but of opposite sign. Hence no net magneto-transverse scattering exists for one Rayleigh scatterer. In fact, Eq. (28) provides the following expression for $F(\theta)$:

$$F_{\mathrm{Rayleigh}}(\theta) = \frac{3mx^3}{4\pi^2(m^2+2)^2} \cos\theta \sin\theta. \tag{29}$$

As the size of the sphere enlarges, the magneto corrections become asymmetric, as seen in Fig. 4. When the size is further increased, new lobes start to appear in the magneto cross section, corresponding to higher spherical harmonics. These lobes do seem to have a net magneto-transverse scattering.

One single Rayleigh scatterer does not induce a net magneto-transverse flux. It is instructive to consider the next simplest case, namely, two Rayleigh scatterers positioned at $\mathbf{r}_1$ and $\mathbf{r}_2$. If their mutual separation well exceeds the wavelength, it is easy to show that the collective cross section simply equals the one-particle cross section multiplied by an interference factor $S(\mathbf{k}, \mathbf{k}') = |\exp[i(\mathbf{k} - \mathbf{k}') \cdot \mathbf{r}_1] + \exp[i(\mathbf{k} - \mathbf{k}') \cdot \mathbf{r}_2]|^2$. This interference factor changes the angular profile of the scattering cross section and ensures that a net magneto-transverse flux remains. The estimate for two Rayleigh particles with an incident wave vector along the interparticle axis is found to be

$$\eta \sim \frac{V_0 B}{k} x^3 \left(\frac{\sin(kr_{12})}{kr_{12}}\right)^2, \qquad \text{if } kr_{12} \gg 1. \tag{30}$$

This simple model suggests that the PHE is in fact a phenomenon generated by interference of different light paths. In Fig. 6 we show how differential cross sections of two particles change in a magnetic field. More scatter-

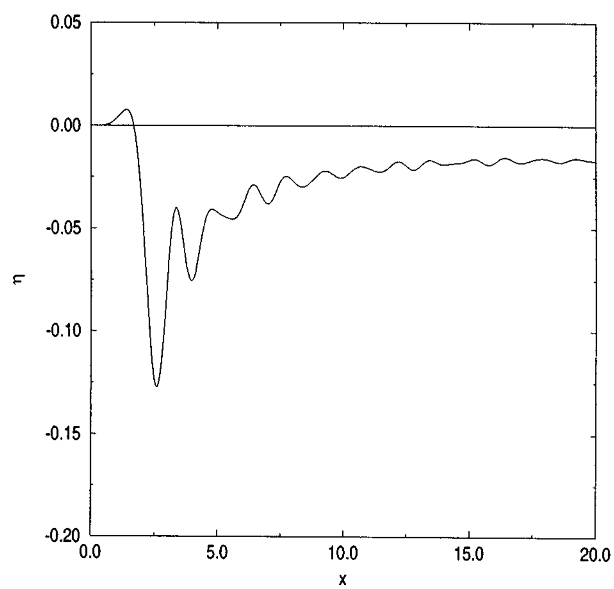

Fig. 5. Normalized magneto-transverse light current η as a function of size parameter x for an index of refraction $m = 1.0946$. The curve is displayed in units of W.

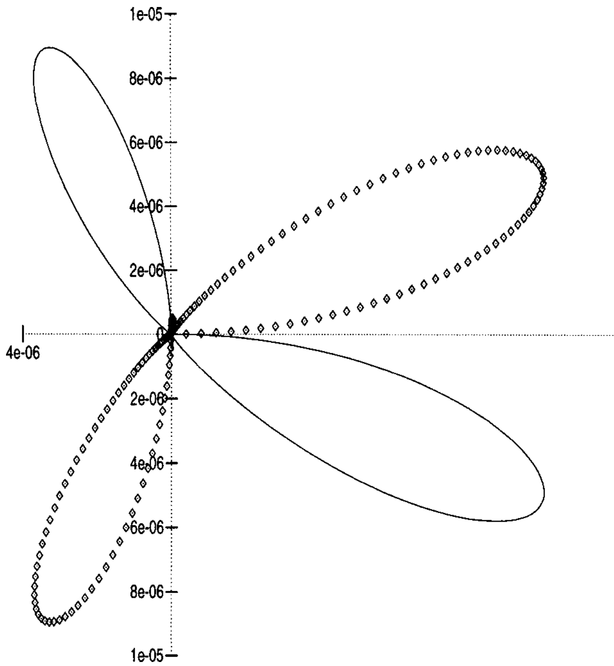

Fig. 6. Magneto-cross section for two Rayleigh scatterers each of size parameter $ka = 0.1$ and separated by a distance corresponding to size parameter $kr_{12} = 5$. In this case the enhanced forward scattering leads also to a net magneto-transverse current along the vertical axis. Axis numbers in the format 4e-06 are equivalent to 4×10^{-6}.

ing is now directed into the forward direction, and as a result the cancellation of the net magneto-transverse flux in Fig. 3 is removed. One Mie sphere qualitatively mimics this simple model and should, on the basis of the principle outlined above, exhibit a magneto-transverse current. The model also suggests that the regime of Rayleigh–Gans scattering[8]—the Born approximation for one sphere that, while contrary to Rayleigh scattering, still allows interferences of different scattering events—should exhibit a net magneto-transverse flux. Indeed, explicit calculations in this regime confirm this statement, with $\eta \sim x^5$, independent of the index of refraction m of the sphere.

B. Geometrical Optics

In the regime of large size parameters, the Mie solution can be obtained from ray optics. Apart from Fraunhofer diffraction (which persists for any finite geometry), the Mie solution for a ray with central impact should be equivalent to the solution for a slab geometry. In our magneto-optic approach this means studying the Faraday rotation in a Fabry–Perot cavity. This model is of special interest because the Fabry–Perot cavity is known to enhance the Faraday rotation[10,11]; i.e., the rotation is additive in the total traversed path length, in contrast to the case of rotary power.

A ray with central impact is characterized by $J = 1$ in Mie theory. We assume that $x \gg 1$ and $y \gg 1$, which allows some simplification in the expression of the Mie coefficients. We find for the Mie coefficients c_1 and d_1 the following behavior:

$$c_1 = \frac{2 \exp[i(x - y)]}{(m + 1)[1 + r \exp(-2iy)]},$$

$$d_1 = \frac{2 \exp[i(x - y)]}{(m + 1)[1 - r \exp(-2iy)]}. \qquad (31)$$

In this formula $r = (m - 1)/(m + 1)$ is the complex Fresnel reflection coefficient. Putting this expression into Eqs. (13), (14), and (21), we can compute the exact behavior of the T matrix in the forward direction. We note with $\mathbf{T}^0_{\text{scatt}}$ the part of $\mathbf{T}^0$ that is due to scattering only, which is obtained from Eq. (15) by replacing the Mie coefficients a_J and b_J with $a_J - 1/2$ and $b_J - 1/2$, since the terms $1/2$ are associated with the Fraunhoffer diffraction, which does not exist for the slab geometry.

Since for a small perturbation we have

$$\mathbf{T} = \mathbf{T}^0_{\text{scatt}} + \mathbf{T}^1 \simeq \mathbf{T}^0_{\text{scatt}} \exp(\mathbf{T}^1/\mathbf{T}^0_{\text{scatt}}), \qquad (32)$$

we see that the change in phase $\delta\phi$ that is due to the magnetic field is in fact related to the imaginary part of $\mathbf{T}^1/\mathbf{T}^0_{\text{scatt}}$. This change in phase can be interpreted as the Faraday effect.

From Eq. (31) we find in the basis of circular polarization

$$\operatorname{Im}\left(\frac{T^1}{T^0_{\text{scatt}}}\right)_{\sigma\sigma'} = \delta\phi(-\sigma)\hat{\mathbf{B}} \cdot \hat{\mathbf{k}}\delta_{\sigma\sigma'}, \qquad (33)$$

with

$$\delta\phi = 2aV_0B\,\frac{1 + R}{1 - R}\,\frac{1}{[1 + \mathscr{M}\sin(2y)^2]}, \qquad (34)$$

with $\mathscr{M} = 4R/(1 - R)^2$ and reflectivity $R = r^2$. We note that the quantity $(-\sigma)\hat{\mathbf{B}} \cdot \hat{\mathbf{k}}$ is conserved for a given ray, which generates the accumulation of the Faraday rotation. The function $\delta\phi$ tends to $2aV_0B$ as $R \to 0$, since it represents the normal Faraday rotation in an isotropic medium of length $2a$, as it should be for our geometry. When R is large, two new factors come into play: $(1 + R)/(1 - R)$, the maximum gain factor of the Faraday rotation, which is due the multiple interference in the Fabry–Perot cavity, and

$$\mathscr{A}(y) = \frac{1}{1 + \mathscr{M}\sin(2y)^2},$$

which is an Airy function of width $4/\sqrt{\mathscr{M}}$. The finesse of the cavity is then $\mathscr{F} = \pi\sqrt{\mathscr{M}}/2$. At resonance, the Faraday rotation is maximally amplified—if we assume no losses—relative to single-path Faraday rotation.[10,11] We stress that one needs $\delta\phi \ll 1$, for Eq. (32) to apply. The Faraday rotation has the effect of splitting each transmission peak in the Fabry–Perot cavity into two peaks of smaller amplitude, with each of the split peaks associated with a different state of helicity.

The amplification of the Faraday rotation is a consequence of the amplified path length of the light. In other words, the Faraday rotation measures the time of interaction of the light with the magnetic field.[12] This time is found to be the dwell time τ of the light in the cavity for this one-dimensional problem. The change in phase follows the simple relation

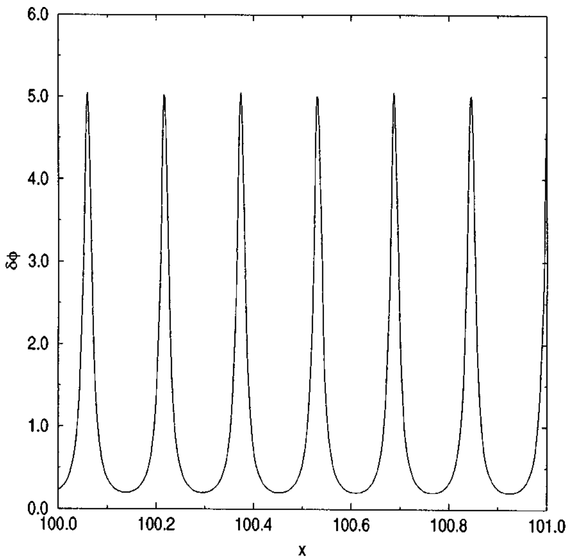

Fig. 7. Magnetically induced change of phase $\delta\phi$—similar to Fabry–Perot modes of a cavity—as a function of size parameter x for the partial wave of $J = 1$, the one with central impact. The curve is for $m = 10$ and has been normalized by the value $2aV_0B$.

$$\delta\phi = V_0 B(\tau c_0/m), \tag{35}$$

where c_0/m is recognized as the speed of light in the sphere.

The dwell time of the light in the cavity varies between a maximum value of

$$\tau_{\text{dwell}}^{\max} = (1 + m^2)a/c_0$$

and a minimum value of

$$\tau_{\text{dwell}}^{\min} = 4m^2/(1 + m^2)a/c_0 .$$

These typical oscillations are visible in the plot of the change of phase $\delta\phi$ of Fig. 7.

6. SUMMARY AND OUTLOOK

In this paper we have addressed the Faraday effect inside a dielectric sphere. We have shown that this theory is consistent with former results concerning the predictions of the light scattered by Rayleigh scatterers in a magnetic field. It is possible to get from this perturbative theory quantitative predictions concerning the Photonic Hall Effect for one single Mie sphere, such as the scattering cross section and the dependence on the size parameter or on the index of refraction.

We will begin experiments that address single Mie scattering in a magnetic field. A second challenge is to incorporate our Mie solution into a multiple-scattering theory.

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