Examen

I – Télégraphe :

a) $\frac{1}{2}\left(1+e^{-2\gamma(t_2-t_1)}\right)$ = proba. de rester dans le même état. 0,5

$\frac{1}{2}\left(1-e^{-2\gamma(t_2-t_1)}\right)$ = proba de quitter l'état.

b)
$$\sum_{n_2} p(n_2,t_2|n_1,t_1) = \sum_{n_2} \frac{1}{2}\left(1+e^{-2\gamma(t_2-t_1)}\right)\delta_{n_1,n_2} + \frac{1}{2}\left(1-e^{-2\gamma(t_2-t_1)}\right)\delta_{n_1,-n_2}$$
$$= \frac{1}{2}\left(1+e^{-2\gamma(t_2-t_1)}\right) + \frac{1}{2}\left(1-e^{-2\gamma(t_2-t_1)}\right)$$
$$= 1.$$
0,5

c)
$$\sum_{n'} p(n_2,t_2|n',t')\,p(n',t'|n_1,t_1).$$

$$= \sum_{n'} \left[\frac{1}{2}\left(1+e^{-2\gamma(t_2-t')}\right)\delta_{n_2,n'} + \frac{1}{2}\left(1-e^{-2\gamma(t_2-t')}\right)\delta_{n_2,-n'}\right]$$
$$\left[\frac{1}{2}\left(1+e^{-2\gamma(t'-t_1)}\right)\delta_{n',n_1} + \frac{1}{2}\left(1-e^{-2\gamma(t'-t_1)}\right)\delta_{n',-n_1}\right]$$

$$= \frac{1}{4}\left(1+e^{-2\gamma(t_2-t')}\right)\left(1+e^{-2\gamma(t'-t_1)}\right)\delta_{n_2,n_1} +$$
$$\frac{1}{4}\left(1-e^{-2\gamma(t_2-t')}\right)\left(1+e^{-2\gamma(t'-t_1)}\right)\delta_{n_2,-n_1} +$$
$$\frac{1}{4}\left(1 \blacksquare e^{-2\gamma(t_2-t')}\right)\left(1 \blacksquare e^{-2\gamma(t'-t_1)}\right)\delta_{n_2,-n_1} +$$
$$\frac{1}{4}\left(1-e^{-2\gamma(t_2-t')}\right)\left(1-e^{-2\gamma(t'-t_1)}\right)\delta_{n_2,n_1}$$

$$= \frac{1}{4}\Big(1+e^{-2\gamma(t_2-t')}+e^{-2\gamma(t'-t_1)}+e^{-2\gamma(t_2-t_1)} +$$
$$1 \blacksquare e^{-2\gamma(t_2-t')} \blacksquare e^{-2\gamma(t'-t_1)}+e^{-2\gamma(t_2-t_1)}\Big)\delta_{n_1,n_2}$$
$$+\frac{1}{4}\Big(1-e^{-2\gamma(t_2-t')}+e^{-2\gamma(t'-t_1)}-e^{-2\gamma(t_2-t_1)}$$
$$+1+e^{-2\gamma(t_2-t')} \blacksquare e^{-2\gamma(t'-t_1)} \blacksquare e^{-2\gamma(t_2-t_1)}\Big)\delta_{n_1,-n_2}$$

$$= \frac{1}{2}\left(1+e^{-2\gamma(t_2-t_1)}\right)\delta_{n_1,n_2} + \frac{1}{2}\left(1-e^{-2\gamma(t_2-t_1)}\right)\delta_{n_1,-n_2}\,.$$ 1,5

d) On a :

$$\dot{p}_1 = \sum_{n} \omega_{n_1 n}\, p_n - \sum_{n} \omega_{n n_1}\, p_n$$

$$= \omega_{n_1 n_2}\, p_2 - \omega_{n_2 n_1}\, p_1$$

or $\omega_{n_1 n_2} = \lim_{\Delta t \to 0} \frac{1}{2}\frac{[2\gamma \Delta t]}{\Delta t} = \gamma = \omega_{n_2 n_1}$.

1

$$\Rightarrow \begin{cases} \dot{p}_1 = \gamma\,(p_2 - p_1) \\ \dot{p}_2 = \gamma\,(p_1 - p_2) \end{cases}$$

e) $\dot{p}_1 = -\gamma p_1 + \gamma p_{-1}$ avec $p_1 + p_{-1} = 1$

$\dot{p}_{-1} = -\gamma p_{-1} + \gamma p_1$

$\Rightarrow \dot{p}_1 = -\gamma p_1 + \gamma\,[1 - p_1] = -2\gamma p_1 + \gamma$.

$\Rightarrow p_1(t) = \lambda e^{-2\gamma t} + \frac{1}{2}$.

or $p_1(t=0) = 1 \Rightarrow \lambda + \frac{1}{2} = 1 \Rightarrow \lambda = \frac{1}{2}$.

$$\begin{cases} p_1(t) = \frac{1}{2}\left(1+e^{-2\gamma t}\right) \\ p_2(t) = 1 - p_1(t) = \frac{1}{2}\left(1-e^{-2\gamma t}\right) \end{cases}$$ 1

$$p_n(t) = \frac{1}{2}\left(1+e^{-2\gamma t}\right)\delta_{n,1} + \frac{1}{2}\left(1-e^{-2\gamma t}\right)\delta_{n,-1}$$ 0,5

f) $p(n,t) = \frac{1}{2}\left(1+e^{-2\gamma t}\right)\delta_{n,1} + \frac{1}{2}\left(1-e^{-2\gamma t}\right)\delta_{n,-1}$.

$$p(n_2,t_2) = \sum_{n_1} p(n_2,t_2|n_1,t_1)\, p(n_1,t_1)$$

$$= \sum_{n_1}\left[\frac{1}{2}\left(1+e^{-2\gamma(t_2-t_1)}\right)\delta_{n_1,n_2} + \frac{1}{2}\left(1-e^{-2\gamma(t_2-t_1)}\right)\delta_{n_1,-n_2}\right]$$
$$\left[\frac{1}{2}\left(1+e^{-2\gamma t_1}\right)\delta_{n_1,1} + \frac{1}{2}\left(1-e^{-2\gamma t_1}\right)\delta_{n_1,-1}\right]$$

$$= \frac{1}{4}\left[1+e^{-2\gamma(t_2-t_1)} + e^{-2\gamma t_1} + e^{-2\gamma t_2}\right]\underbrace{\delta_{n_1,n_2}\,\delta_{n_1,1}}_{\delta_{n_2,1}}$$
$$+\frac{1}{4}\left[1-e^{-2\gamma(t_2-t_1)} + e^{-2\gamma t_1} - e^{-2\gamma t_2}\right]\underbrace{\delta_{n_1,-n_2}\,\delta_{n_1,1}}_{\delta_{n_2,-1}}$$
$$+\frac{1}{4}\left[1+e^{-2\gamma(t_2-t_1)} - e^{-2\gamma t_1} - e^{-2\gamma t_2}\right]\underbrace{\delta_{n_1,n_2}\cdot\delta_{n_1,-1}}_{\delta_{n_2,-1}}$$
$$+\frac{1}{4}\left[1-e^{-2\gamma(t_2-t_1)} - e^{-2\gamma t_1} + e^{-2\gamma t_2}\right]\underbrace{\delta_{n_1,-n_2}\,\delta_{n_1,-1}}_{\delta_{n_2,1}}$$

$$= \frac{1}{4}\left[2+2e^{-2\gamma t_2}\right]\delta_{n_2,1} + \frac{1}{4}\left[2-2e^{-2\gamma t_2}\right]\delta_{n_2,-1}.$$

$$= \frac{1}{2}\left[1+e^{-2\gamma t_2}\right]\delta_{n_2,1} + \frac{1}{2}\left[1-e^{-2\gamma t_2}\right]\delta_{n_2,-1}.$$

ok.

1

II- Potentiel harmonique.

a) $\langle \Gamma(t) \rangle = 0$ moyenne nulle, pas de direction privilégiée (0,5)

$\langle \Gamma(t)\Gamma(t') \rangle$ pas de mémoire du bruit.

impatance $\Rightarrow$ Markov.

b) régime visqueux : $\left| m\gamma \frac{dx}{dt} \right| \gg \left| m \frac{d^2x}{dt^2} \right|$

$$\Rightarrow 0 = - m\gamma \frac{dx}{dt} - \frac{dU(x)}{dx} + m\Gamma(t)$$

$$\Rightarrow \boxed{\frac{dx}{dt} = - \frac{1}{m\gamma} \frac{dU(x)}{dx} + \frac{1}{\gamma} \Gamma(t)}$$ (0,5)

c) On a : $\frac{dp(x,t)}{dt} = -\frac{\partial}{\partial x}\left[a_1 p(x,t)\right] + \frac{\partial^2}{\partial x^2}\left[a_2 p(x,t)\right]$.

$$a_1(x,t) = \int dx' \, (x'-x) \, p(x',t'|x,t)$$ (1)

$$a_2(x,t) = \frac{1}{2}\int dx' \, (x'-x)^2 p(x',t'|x,t)$$

d) On a : $x(t+\Delta t) - x(t) = \int_t^{t+\Delta t} -\frac{1}{m\gamma} \frac{dU(x)}{dx} + \int_t^{t+\Delta t} \frac{1}{\gamma} \Gamma(t) \, dt$.

$$= -\frac{\Delta t}{m\gamma} \frac{dU(x)}{dx} + \frac{1}{\gamma} \int_{t+\Delta t}^{t} \Gamma(t) \, dt$$

$$\Rightarrow \langle x(t+\Delta t) - x(t) \rangle = -\frac{\Delta t}{m\gamma} \frac{dU(x)}{dx}.$$

et $\boxed{a_1(x,t) = -\frac{1}{m\gamma} \frac{dU(x)}{dx}}$ (0,5)

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$$[x(t+\Delta t)-x(t)]^2 = \Big[\underbrace{\int_t^{t+\Delta t} dt' \Big(-\frac{1}{m\gamma}\frac{dU(x)}{dx}\Big)}_{\to \Delta t^2} + \underbrace{\int_t^{t+\Delta t} \frac{1}{\gamma}\frac{\Gamma(t')}{dt'}}_{\to 0}\Big]^2$$

$$= \int_t^{t+\Delta t} dt' \frac{1}{\gamma^2}\int_t^{t+\Delta t} dt'' \,\Gamma(t')\Gamma(t'')$$

$$\langle \quad \rangle \quad = \frac{1}{\gamma^2}\int_t^{t+\Delta t} dt' \int_t dt'' \; 2D\,\delta(t'-t'')$$

$$= \frac{1}{\gamma^2} \cdot 2D\,\Delta t.$$

$$\Rightarrow \boxed{a_2(x,t) = \frac{D}{\gamma^2}}$$

0,5

et
$$\boxed{\frac{\partial p(x,t)}{\partial t} = \frac{\partial}{\partial x}\Big[\frac{1}{m\gamma}\frac{dU(x)}{dx}\,p(x,t)\Big] + \frac{D}{\gamma^2}\frac{\partial^2 p(x,t)}{\partial x^2}}$$

0,5

II – Potentiel harmonique.

$U(x) = \frac{1}{2}kx^2.$

a) $$\frac{\partial p(x,t)}{\partial t} = \frac{\partial}{\partial x}\Big[\frac{1}{m\gamma}kx\,p\Big] + \frac{D}{\gamma^2}\frac{\partial^2 p}{\partial x^2}.$$

On a :

$$\frac{d}{dt}\langle x(t)^n\rangle = \frac{\partial}{\partial t}\int dx\, x^n p(x,t) = \int dx\, x^n \frac{\partial p(x,t)}{\partial t}$$

$$= \int dx\, x^n \Big\{\frac{\partial}{\partial x}\Big[\frac{1}{m\gamma}kx\,p\Big] + \frac{D}{\gamma^2}\frac{\partial^2 p}{\partial x^2}\Big\}.$$

$$= -\int dx\; n x^{n-1} \frac{1}{m\gamma} k x p + \frac{D}{\gamma^2} n x^{n-1} \frac{\partial p}{\partial x}$$

$$= -\frac{\cancel{n} k}{m\gamma} \langle x^n \rangle - \frac{D n}{\gamma^2} \int x^{n-1} \frac{\partial p}{\partial x}$$

$$= -\frac{k}{m\gamma} n \langle x^n \rangle + \frac{D}{\gamma^2} n(n-1) \int x^{n-2} p$$

$$\boxed{= -\frac{k}{m\gamma} n \langle x^n \rangle + \frac{D}{\gamma^2} n(n-1) \langle x^{n-2} \rangle.}$$ 1

b) $\boxed{n=1}$

$$\frac{d}{dt} \langle x \rangle = -\frac{k}{m\gamma} \langle x \rangle$$

$$\Rightarrow \langle x(t) \rangle = \lambda e^{-\frac{k}{m\gamma} t}.$$

or $p(x, t=0) = \delta(x - x_0)$

$$\Rightarrow \langle x(t=0) \rangle = \int dx\; x\, p(x, t=0) = x_0$$

$$\Rightarrow \lambda = x_0 \text{ et}$$

$$\boxed{\langle x(t) \rangle = x_0 e^{-\frac{k}{m\gamma} t}}$$ 1,5

$$\frac{d}{dt} \langle x^2 \rangle = -\frac{2k}{m\gamma} \langle x^2 \rangle + \frac{2D}{\gamma^2}$$

$$\Rightarrow \langle x^2(t) \rangle = \alpha e^{-\frac{2k}{m\gamma} t} + \frac{2 D\, m\gamma}{\gamma^2\, 2k} = \alpha e^{-\frac{2k}{m\gamma} t} + \frac{D m}{\gamma k}$$

$$\langle x^2(t=0)\rangle = x_0^2 = \alpha + \frac{mD}{\gamma k} \Rightarrow \alpha = x_0^2 - \frac{mD}{\gamma k}$$

$$\Rightarrow \langle x^2(t)\rangle = \left(x_0^2 - \frac{mD}{\gamma k}\right) e^{-\frac{2k}{m\gamma}t} + \frac{mD}{\gamma k}.$$

$$\longrightarrow \frac{mD}{\gamma k}$$

$$\text{a } \left\langle \frac{1}{2} k x^2 \right\rangle = \frac{1}{2} k_B T = \frac{1}{2} \frac{m D k}{\gamma k} = \frac{m D}{2\gamma}$$

0,5

$$\Rightarrow \boxed{k_B T = \frac{m D}{\gamma}}$$

c) $$\frac{dM_n(t)}{dt} = \frac{d}{dt} \int p_n(x,t) \left[x - \langle x(t)\rangle\right]^n$$

$$= \int \frac{dp_n(x,t)}{dt} \left[x - \langle x(t)\rangle\right]^n$$

$$+ \int p_n(x,t)\, n \left[x - \langle x(t)\rangle\right]^{n-1} \left[-\frac{d\langle x(t)\rangle}{dt}\right]$$

$$= \int \frac{\partial}{\partial x}\left[\frac{k}{m\gamma} x p + \frac{D}{\gamma^2}\frac{\partial p}{\partial x}\right]\left[x - \langle x(t)\rangle\right]^n$$

$$- \int p\, n \left[x - \langle x(t)\rangle\right]^{n-1}\left[-\frac{k}{\gamma m}\langle x\rangle\right]$$

$$= -\int \frac{k}{\gamma m} x p\, n \left[x - \langle x(t)\rangle\right]^{n-1} + \int \frac{k n}{\gamma m}\langle x\rangle p \left[x - \langle x\rangle\right]^{n-1}$$

$$+ \int \frac{D}{\gamma^2} p\, n[n-1]\left[x - \langle x\rangle\right]^{n-2}$$

$$= -\frac{kn}{\gamma m}\left[x - \langle x(t)\rangle\right]^n + \frac{D}{\gamma^2} n[n-1]\left[x - \langle x\rangle\right]^{n-2}$$

$$= -\frac{kn}{\gamma m} M_n(t) + \frac{D}{\gamma^2} n(n-1) M_{n-2}(t).$$

1,5

On a i) (1)

$$M_n'(t) = -\frac{kn}{\gamma m} M_n(t) + \frac{D}{\gamma^2} n(n-1) M_{n-2}(t)$$

(8)

$$M_n(t) = \lambda_n(t)\, e^{-\frac{kn}{\gamma m}t}$$

$$M_n'(t) = \lambda_n'(t)\, e^{-\frac{kn}{\gamma m}t} = \frac{D}{\gamma^2} n(n-1) M_{n-2}(t)$$

$$\Rightarrow \lambda_n'(t) = \frac{D}{\gamma^2} n(n-1)\, e^{\frac{kn}{\gamma m}t} M_{n-2}(t)$$

$$\text{et } \lambda_n(t) = \frac{D}{\gamma^2} n(n-1) \int e^{\frac{kn}{\gamma m}t} M_{n-2}(t) + \alpha_n.$$

$$\Rightarrow M_n(t) = \left[\frac{D}{\gamma^2} n(n-1) \int e^{\frac{kn}{\gamma m}t} M_{n-2}(t)\right] e^{-\frac{kn}{\gamma m}t}$$

$$+ \alpha_n\, e^{-\frac{kn}{\gamma m}t}.$$

(1)

• $n=1$ $\quad M_1(t) = 0.$

$$\Rightarrow M_3(t) \propto \alpha_n\, e^{-\frac{k3}{\gamma m}t}.$$

$$\text{et } M_3(t=0) = \langle (x - \langle x\rangle)^3 \rangle = 0.$$

(1)

$$\Rightarrow \boxed{M_{2h+1} = 0}$$

• $n=2$ $\quad M_2(t) = \alpha_2\, e^{-\frac{k2}{\gamma m}t} + \frac{Dm}{\gamma k}$

$$M_2(t=0) = \langle (x - \langle x\rangle)^2 \rangle = \langle x^2 - 2x\langle x\rangle + \langle x\rangle^2 \rangle$$

$$= \langle x^2\rangle - 2\langle x\rangle^2 + \langle x\rangle^2 = \langle x^2\rangle - \langle x\rangle^2.$$

$$\Rightarrow M_2(t=0) = 0$$

$$\Rightarrow \alpha_2 + \frac{Dm}{\gamma k} = 0 \Rightarrow \alpha_2 = \frac{Dm}{\gamma k}$$

$$\boxed{M_2(t) = \frac{Dm}{\gamma k}\left[1 - e^{-\frac{2k}{\gamma m}t}\right]}$$

$$M_4(t) = \frac{D}{\gamma^2}\, 4\times 3 \left\{\int e^{\frac{4k}{\gamma m}t}\, \frac{Dm}{\gamma k}\left[1 - e^{-\frac{2k}{\gamma m}t}\right]\right\} \times e^{-\frac{4k}{\gamma m}t}$$

$$+ \alpha_4 e^{-\frac{4k}{\gamma m}t}$$

$$= \frac{12 D^2 m}{\gamma^3 k}\left[e^{\frac{4k}{\gamma m}t} \times \frac{\gamma m}{4k} - e^{\frac{2k}{\gamma m}t}\,\frac{\gamma m}{2k}\right] e^{-\frac{4k}{\gamma m}t}$$

$$+ \alpha_4 e^{-\frac{4k}{\gamma m}t}$$

$$= \frac{12 D^2 m}{\gamma^3 k}\left[\frac{\gamma m}{4k}\right]\left[1 - 2e^{-\frac{2k}{\gamma m}t}\right]$$

$$+ \alpha_4 e^{-\frac{4k}{\gamma m}t}$$

$$M_4(t=0) = 0 \Rightarrow \frac{12 D^2 m^2}{4k^2\gamma^2} + \alpha_4 = 0 \Rightarrow \alpha_4 = \frac{3D^2 m^2}{k^2\gamma^2}.$$

$$\Rightarrow M_4(t) = \frac{3D^2m^2}{k^2\gamma^2}\left[1 - 2e^{-\frac{2k}{\gamma m}t} + e^{-\frac{4k}{\gamma m}t}\right]$$

$$= \frac{3D^2m^2}{k^2\gamma^2}\left[1 - e^{-\frac{2k}{\gamma m}t}\right]^2$$

ii) $M_{2p}(t) = C_p \, [M_2(t)]^p$

$$\frac{dM_{2p}(t)}{dt} = C_p \, p \, [M_2(t)]^{p-1} \, \frac{dM_2(t)}{dt}$$

et $\frac{dM_2(t)}{dt} = -\frac{2k}{\gamma m} M_2 + \frac{D}{\gamma^2} 2(2-1)$

$$= -\frac{2k}{\gamma m} M_2 + \frac{2D}{\gamma^2}.$$

$$\Rightarrow C_p \, p \, [M_2(t)]^{p-1} \left[-\frac{2k}{\gamma m} M_2 + \frac{2D}{\gamma^2} \right]$$

$$= -\frac{k}{\gamma m} \cdot 2p \, C_p \, [M_2(t)]^p + \frac{D}{\gamma^2} 2p(2p-1) \, C_{p-1} [M_2]^{p-1}$$

$$\Rightarrow C_p \, p \, [M_2(t)]^p \left(-\frac{2k}{\gamma m}\right) + C_p \, p \, [M_2(t)]^{p-1} \frac{2D}{\gamma^2}.$$

$$= C_p \, p \, [M_2(t)]^p \left(-\frac{2k}{\gamma m}\right) + C_{p-1} \, p(2p-1) \, [M_2]^{p-1} \frac{2D}{\gamma^2}.$$

$$\Rightarrow C_p = C_{p-1} \, (2p-1).$$

$$\Rightarrow \boxed{C_p = C_{p-1} \; 2p-1}$$

1,0

$C_1 = C_0$

$C_2 = C_1 \, \frac{3}{2} = \frac{3}{2}$

$C_3 = C_2 \, \frac{5}{3} = \frac{5}{3} \, \frac{3}{2} = \frac{5}{2} \ldots$

$C_4 = C_3 \cdot \frac{7}{4} = \frac{7}{4} \, \frac{5}{2} = \frac{35}{8}$

e) $p(x,t) = e^{-\frac{M(t)}{2}x^2 + N(t)x + \Omega(t)}$

$$\frac{\partial p}{\partial t} = \left[-M'(t)\frac{x^2}{2} + N'(t)x + \Omega'(t)\right]p$$

$$\frac{\partial p}{\partial x} = \left[-Mx + N\right]p.$$

$$\frac{\partial}{\partial x}\left[\frac{kx}{m\gamma}p + \frac{D}{\gamma^2}(-Mx+N)p\right]$$

$$= \frac{k}{m\gamma}\left[p + x\frac{\partial p}{\partial x}\right] + \frac{D}{\gamma^2}\left(-Mp + (-Mx+N)\frac{\partial p}{\partial x}\right)$$

$$= \frac{k}{m\gamma}\left[p + x(-Mx+N)p\right] + \frac{D}{\gamma^2}\left(-Mp + (-Mx+N)^2 p\right)$$

$$= \frac{k}{m\gamma}\left[\underline{p} - \underline{Mx^2p} + \underline{Nxp}\right] + \frac{D}{\gamma^2}\left(\underline{-Mp} + \underline{M^2x^2p} - \underline{2NMxp} + \underline{N^2p}\right).$$

$$= p\left\{\left(\frac{k}{m\gamma} - \frac{DM}{\gamma^2} + \frac{N^2D}{\gamma^2}\right) + x\left(\frac{k}{m\gamma}N - \frac{D2NM}{\gamma^2}\right) + x^2\left(-\frac{k}{m\gamma}M + \frac{D}{\gamma^2}M^2\right)\right\}.$$

$$= p\left\{-M'\frac{x^2}{2} + N'x + \Omega'\right\}.$$

$$\Rightarrow \begin{cases} -\dfrac{M'}{2} = -\dfrac{k}{m\gamma} M + \dfrac{D}{\gamma^2} M^2 \\ N' = \dfrac{k}{m\gamma} N - \dfrac{2D}{\gamma^2} N M \qquad (2) \\ \ell' = \dfrac{k}{m\gamma} - \dfrac{DM}{\gamma^2} + \dfrac{N^2 D}{\gamma^2}. \end{cases}$$

b)

posons: $$\frac{dM^{-1}}{dt} = -M^{-2}\frac{dM}{dt}$$

$$= -M^{-2}\left(-\frac{k}{m\gamma} M + \frac{D}{\gamma^2} M^2\right)$$

$$= \frac{k}{m\gamma} M^{-1} - \frac{D}{\gamma^2}$$

$$\Rightarrow M^{-1}(t) = \lambda e^{\frac{k}{m\gamma} t} - \frac{D}{\gamma^2}\frac{m\gamma}{k}$$

$$= \lambda e^{\frac{k}{m\gamma} t} - \frac{mD}{k\gamma}.$$

- $$M^{-1}(t=0) = 0 = \lambda - \frac{mD}{k\gamma}$$

$$\Rightarrow M^{-1}(t) = \frac{mD}{k\gamma}\left[e^{\frac{k}{m\gamma} t} - 1\right].$$

$$M(t) = \frac{k\gamma}{mD} \frac{1}{e^{\frac{kt}{\gamma m}} - 1}. \qquad (1)$$

III- Diffusion

a) On a $p_0(x,t|x_0) = \frac{1}{\sqrt{4\pi D t}} e^{-\frac{(x-x_0)^2}{4Dt}}$. 1

b) $\frac{\partial p}{\partial t} = D \frac{\partial^2 p}{\partial x^2} - r\, p(x,t|x_0) + r\, \delta(x-x_0)$ 0,5

c) $\frac{\partial p(x,t)}{\partial t} = \int \frac{e^{-ikx}}{\sqrt{2\pi}} \frac{\partial \tilde{p}(k,t)}{\partial t} dk$

$$\frac{\partial^2 p(x,t)}{\partial x^2} = \int \frac{e^{-ikx}}{\sqrt{2\pi}} (-k^2)\, \tilde{p}(k,t)\, dk.$$

$$\delta(x-x_0) = \int e^{-ikx} \frac{e^{ikx_0}}{2\pi} dk.$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}} \frac{\partial \tilde{p}}{\partial t} = \frac{D}{\sqrt{2\pi}}(-k^2)\tilde{p} - \frac{r}{\sqrt{2\pi}}\tilde{p} + \frac{r}{2\pi} e^{+ikx_0} = 0$$

$$\Rightarrow -\left[k^2 D + r\right]\tilde{p} = -r \frac{e^{ikx_0}}{\sqrt{2\pi}}$$

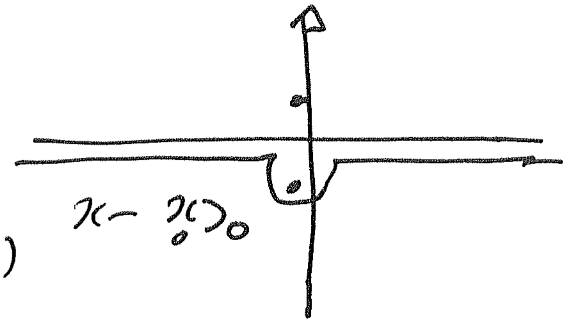
$$\Rightarrow \boxed{\tilde{p}(k,t) = \frac{1}{\sqrt{2\pi}} \frac{r\, e^{ikx_0}}{k^2 D + r}.}$$

$$p(x,t) = \int \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2\pi}} \frac{(+r)\, e^{ikx_0}}{k^2 D + r} \cdot e^{-ikx}\, dk$$

$$= + \frac{r}{2\pi} \int \frac{e^{ik(x_0 - x)}}{k^2 D + r}\, dk.$$

- $k^2 D + r = 0$

$k^2 = -\frac{R}{D} \qquad k = \pm i\sqrt{\frac{R}{D}}.$



$$p(x,t) = +\frac{R}{2\pi}\int_{-\infty}^{+\infty} \frac{e^{ik(x_0-x)}}{D\left(k+i\sqrt{\frac{R}{D}}\right)\left(k-i\sqrt{\frac{R}{D}}\right)}$$

Pour $x>0$: $k = -i\sqrt{\frac{R}{D}}$

$$p(x,t) = \frac{R}{2\pi}\, 2\pi i \, \frac{e^{i\left(-i\sqrt{\frac{R}{D}}\right)(x_0-x)}}{-2i\sqrt{\frac{R}{D}}\,D} \qquad x - x_0 > 0$$

$x<0$: $k = i\sqrt{\frac{R}{D}}$

$$p(x,t) = +\frac{R}{2\pi}(-2\pi i)\, \frac{e^{i\left(i\sqrt{\frac{R}{D}}\right)(x_0-x)}}{2i\sqrt{\frac{R}{D}}\,D} \qquad x - x_0 < 0.$$

$$= \frac{1}{2}\sqrt{\frac{R}{D}}\; e^{\sqrt{\frac{R}{D}}(x_0-x)} \qquad x - x_0 > 0$$

$$= \frac{1}{2}\sqrt{\frac{R}{D}}\; e^{-\sqrt{\frac{R}{D}}(x_0-x)} \qquad x - x_0 < 0.$$

$$\boxed{p(x,t) = \frac{\alpha_0}{2}\, e^{-\alpha_0 |x - x_0|}}$$

2,5

(12)

d) $\underbrace{Q(0,t)=0}$ et $\underbrace{Q(x,0)=1}$

probà. de ne pas avoir atteint l'énergie au temps t étant partie de x à t=0

proba. de ne pas avoir atteint l'énergie à t étant partie de x à t=0.

$$q(x,s) = \int_0^\infty dt\, e^{-st} Q(x,t)$$

$$\bullet \int_0^\infty dt\, e^{-st} \frac{\partial Q}{\partial t} = D \int_0^\infty e^{-st} \frac{\partial^2 Q}{\partial x^2} - r \int_0^\infty e^{-st} Q(x,t)$$
$$+ r \int_0^\infty e^{-st} Q(x_0,t)$$

$$\Rightarrow \left[e^{-st} Q\right] - \int_0^\infty dt\, (-s)\, Q\, e^{-st}$$
$$= D \int_0^\infty e^{-st} \frac{\partial^2 Q}{\partial x^2} - r \int_0^\infty e^{-st} Q(x,t)$$
$$+ r \int_0^\infty e^{-st} Q(x_0,t)$$

$$1 +$$
$$\Rightarrow s\, q(x,s) = D \frac{\partial^2 q(x,s)}{\partial x^2} - r\, q(x,s) + r\, q(x_0,s).$$

$$\Rightarrow \boxed{D \frac{\partial^2 q(x,s)}{\partial x^2} - (r+s)\, q(x,s) = -1 - r\, q(x_0,s)}$$

1,5

e) $D \frac{\partial^2 q(x,s)}{\partial x^2} - (r+s)\, q(x,s) = 0$

$$\Rightarrow \quad q'' - \frac{(r+s)}{D}\, q(x,s) = 0$$

$$\boxed{q(x,s) = A e^{\alpha x} + B e^{-\alpha x} + \frac{1 + r\, q(x_0,s)}{r+s}}$$

$$\hookrightarrow \quad q(x,s) = B e^{-\alpha x} + \frac{1 + r\, q(x_0,s)}{r+s}$$

• $Q(0,t) = 0$

$$\Rightarrow \quad q(0,s) = \int_0^\infty dt\, e^{-st}\, Q(0,t) = 0.$$

$$\Rightarrow \quad B + \frac{1 + r\, q(x_0,s)}{1+s} = 0.$$

$$\Rightarrow \quad q(x_0,s) = -\left(\frac{1 + r\, q(x_0,s)}{r+s}\right) e^{-\alpha x_0} + \frac{1 + r\, q(x_0,s)}{r+s}$$

$$\Rightarrow \quad (r+s)\, q(x_0,s) = -(1 + r\, q(x_0,s))\, e^{-\alpha x_0} + 1 + r\, q(x_0,s)$$

$$\boxed{q(x_0,s) = \frac{1 - e^{-\alpha x_0}}{s + r e^{-\alpha x}}}$$

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f) $T(x_0) = -\int_0^\infty dt\, t\, \frac{\partial Q(x_0,t)}{\partial t}$

$$= \left[t\, Q(x_0,t) \right]_0^\infty + \int_0^\infty dt\, Q(x_0,t)$$

$$= \int_0^\infty dt\, Q(x_0,t)$$

$$= q(x_0, s=0)$$

$$\Rightarrow \boxed{T(x_0) = \frac{1-e^{-\alpha x_0}}{R\, e^{-\alpha x_0}} = \frac{e^{\alpha x_0}-1}{R}}$$ 1

g) 1