Finite size scaling in first and second order phase transitions

Références

[1] Principles of condensed matter physics, P. M. Chaikin and T. C. Lubensky, Cambridge Univ. Press

1 Critical exponents in mean-field theory

In Landau theory, the density of free energy for a magnetic system in a magnetic field h takes the form

$$f(m) = f_0 + \frac{r}{2}m^2 + um^4 - mh,$$

where m denotes the magnetization, r is of the form $a(T - T_c)$, and u is another coefficient independent of temperature. The magnetization is first assumed to be uniform spatially and the system is of infinite size. Such an expansion is particularly meaningful to describe the behavior of the system near its critical point.

▷ **1-1** Show that the magnetization near the critical point behaves as $m \simeq (T - T_c)^\beta$, where β is a critical exponent.

▷ **1-2** Investigate under the same conditions, the behavior of the magnetic susceptibility and that of the specific heat. Find also the behavior of the order parameter at the critical point as a function of the magnetic field.

▷ **1-3** We now consider a space dependent magnetization. Generalize the Landau free energy to take into account the energy cost of having a space dependent magnetization. Derive from this the behavior of the magnetic suspectibility as function of wave vector then as function of space. Calculate the critical exponent ν.

▷ **1-4** Show that the hyperscaling relation $\gamma + 2\beta = d\nu$ only holds for a specific dimension. What is meaning of "universality class" in this context ? Why are the values of the critical exponents which are measured in real systems often different from the values calculated with the Landau model ?

2 Finite size scaling near a continuous phase transition

▷ **2-1** Explain why in practice the divergences of some of the quantities introduced above for an infinite system can not be observed in practice in a simulation. What is the behavior of the correlation length for a system of size L at the critical point ?

▷ **2-2** Within a scaling approach, justify the following relation :

$$\frac{C_L(0)}{C(t)} = F\left(\frac{\xi_L(0)}{\xi_\infty(t)}\right),$$

where F is a scaling function, $C_L(0)$ is the value of the specific heat at the critical point for a system of size L.

▷ **2-3** Assuming that the ratio $\frac{\xi_L(0)}{\xi(t)}$ is finite and independent of L, deduce that $t \simeq L^x$ where x is an exponent to be determined.

▷ **2-4** Show that $C_L(0) \simeq L^y$, where y is to be determined.

▷ **2-5** Using the same method, show that the magnetic susceptibility $\chi \simeq L^z$.

3 Finite size scaling near a discontinuous phase transition

We consider the Ising model in dimension d in a magnetic field h. Near $h = 0$, the system can find itself in either a magnetization $m = m_0$ or $m = -m_0$.

▷ **3-1** Show graphically the magnetization as function of h, for an infinite and for a finite system.

▷ **3-2** Express the free energy near $m = m_0$ or $m = -m_0$ as function of the susceptibility χ, m and m_0. Deduce from this an approximate expression for the probability to find the value m for a finite system of size L in the form of the sum of two gaussian distributions.

▷ **3-3** Deduce from this the magnetic susceptibility for a finite system χ_L as function of the susceptibility for the infinite system χ. Compare this result with the case of a continuous transition.

▷ **3-4** Show that this solution satisfies the rather general relation :

$$\langle m^2 \rangle_L - \langle m \rangle_L^2 = k_B T \frac{\chi_L}{L^d},$$

where the index L indicates the finite size of the system.

▷ **3-5** What is the relative weights of the two gaussian distributions in the calculation above ?

We shall prove that a result of this kind is very general and follow from basic assumptions about the symmetries of the model. To do so, let us consider now a discrete spin model with hamiltonian

$$H_N(\sigma, B) = H_N(\sigma, 0) - BM_N(\sigma),$$

where σ denotes the spin variables, B the magnetic field and M_N is the magnetization for a system of size N.

Prove the following fluctuation relation

$$P_B(M) = P_B(-M)e^{2\beta BM},$$

and interpret the result.